VOL. XXVII. No. 2. APRIL, 1917

THE MONIST

A Quarterly Magazine

Devoted to the Philosophy of Science

Founded by EDWARD C. HEGELER

CONTENTS:

PAGE

THE TEXT OF THE RESURRECTION IN MARK, AND ITS TESTIMONY TO THE APPARITIONAL THEORY.
ALBERT J. EDMUNDS 161

BERGSONISM IN ENGLAND.
J. W. SCOTT 179

THE PRESENT STATUS OF THE UNCONSCIOUS.
GUSTAVE A. FEINGOLD 205

NIRVANA (Poem).
PAUL CARUS 233

THE MANUSCRIPTS OF LEIBNIZ ON HIS DISCOVERY OF THE DIFFERENTIAL CALCULUS. Part II.
J. M. CHILD 238

CRITICISMS AND DISCUSSIONS.

Mechanism and the Problem of Freedom. GERTRUDE CARMAN BUSSEY... 295

Determinism of Free Will. PAUL CARUS 306

The Belief in God and Immortality (LEUBA) 311

Sir Oliver Lodge on Life After Death 316

Current Periodicals 319

CHICAGO
THE OPEN COURT PUBLISHING COMPANY
1917

UMI

COPYRIGHT BY
THE OPEN COURT PUBLISHING COMPANY
1917

VOL. XXVII APRIL, 1917 NO. 2

THE MONIST

THE TEXT OF THE RESURRECTION IN MARK, AND ITS TESTIMONY TO THE APPARITIONAL THEORY.

WITH A PREFACE ON LUKE'S MUTILATION OF MARK.

THE greatest literary problem in the New Testament is: What is the matter with the Gospel of Mark? Something happened to the end of it in the first or second century, and for ages thereafter it was left truncated in the middle of a sentence or else supplied with a shorter conclusion than the present one, which scholars long kept to themselves. Edwin A. Abbott, however, gave it in his forgotten Gospel analysis of 1884, and the Nonconformist translators of *The Twentieth Century New Testament* have also given it; but it does not appear in any official translation, though the Revised Version mentions it in a note at Mark xvi. 8. This is the note:

"The two oldest Greek manuscripts, and some other authorities, omit from verse 9 to the end. *Some other authorities have a different ending to the Gospel.*"

Here is the "different ending," translated from a ninth-century manuscript in the National Library of France, Codex L, which gives both conclusions, but puts this one first. (We prefix to it the connecting words of Mark):

II. *And they went out and fled from the sepulcher, for*

trembling and astonishment had come upon them; and they said nothing to any one, for they were afraid of.

* * * * * * * * * * * * *

[Thirteen ornamental marks.]

Where also you must give currency to this:

Now, all things that were commanded, they showed forth in few words unto those about Peter. And after these things Jesus himself, also, from the East even unto the West, sent forth through them the holy and incorruptible preaching of eternal salvation.

But there is also current the following, after the words: FOR THEY WERE AFRAID OF:

Now, when he was risen early etc. (as in our common versions, Mark xvi. 9-20).

In their Introduction to the New Testament (Cambridge, 1881, pp. 298, 299) Westcott and Hort remark on the above "less known alternative supplement" to Mark: "In style it is unlike the ordinary narrative of the Evangelists, but comparable to the four introductory verses of St. Luke's Gospel." Conybeare, in his great book, *Myth, Magic and Morals,* throws out the suggestion that Luke mutilated the first edition of Mark because he disagreed with its contents: viz., an account of apparitions in Galilee, whereas he expressly limits all these phenomena to Judea, by making Jesus order the apostles to stay in Jerusalem until Pentecost. (Luke xxiv. 49; Acts i. 4). If Luke mutilated Mark, then why not go further and say that he wrote this smooth-flowing supplement to round him out? The word συντομως, "in few words," is never found in the New Testament except in this shorter Mark Appendix and in Luke's Acts of the Apostles (xxiv. 4).

Ἐξαγγελλω, "to show forth," also occurs only in the Pauline or Lucan (for Luke was Paul's secretary) Epistle of Peter (1 Pet. ii. 9). Ἐξαποστελλω, "to send forth," is used seven times in Acts, thrice in Luke's Gospel, and once

by his master Paul. "Incorruptible" occurs only in Paul and the Pauline 1 Peter.

Luke represents the aristocratic tradition of the capital, which said: "It all happened here!" Mark represents the rural tradition of Galilee, which said: "Our poor parish was the scene of these wonderful things!" So the young man in white, in Mark, says at the sepulcher: *Go, tell his disciples and Peter: Behold, I am going to Galilee ahead of you. There shall ye see me.* (Thus read some of our best manuscripts, in the first person.)

Another thing: Luke and John both make the apparitions real. In these later Gospels Jesus is objective after the Resurrection: he eats broiled fish in Luke, while in John the wounds in his hands and side are felt by Thomas. Now as the earliest account of the Resurrection in Paul (1 Cor. xv. 4-8) makes the event a series of apparitions, it is probable that the second earliest account, Mark's, did the same. Indeed in Matthew xxviii. 17 (under suspicion of being taken from the lost ending of Mark), "some doubted." This was because some saw the figure and others did not. Luke and John leave no room for doubt: the evidence is sensuous, not subjective.

The first Christian heresy was Docetism, the belief that Jesus even in life was a phantom. His flesh and blood were unreal; he did not really suffer; his bodily functions were different from human ones or even non-existent. To fight this heresy the First Epistle of John was written, and a curse pronounced upon those who doubted that Jesus had been actual flesh and blood (1 John iv. 2, 3). Consequently if Mark repeated Paul's impression that the Galilean apparitions were the same in kind as the one to himself on the Damascus road, then Mark must go. Who was the likeliest one to do this work of excision? Answer: Luke. He was the most literary of all the Evangelists. He is

the only one of them who says "I." Moreover, as Harnack has pointed out, he betrays an animus against Mark, animadverting upon his conduct in Acts xv. 36-41. In his own Gospel Prologue, Luke is undoubtedly thinking of him as one of the "many" who have "undertaken" to write the life of Jesus, but who have not begun "accurately from the first" nor set forth "in order" the sayings and events. Add to this the Jerusalem tradition of the Resurrection against the Galilean, and the flesh-and-blood appearances against the phantom who is only to be "seen" ("there shall ye see me," in Mark), and we have motive enough for Luke's high-handed act.

Indeed, we can even surmise the reason why he made the excision in the middle of a sentence. He would hardly do this except to get rid of an offensive word. If Mark had read:

They said nothing to any one, for they were afraid of the apparition,

this last word would have been the red rag. There must be no apparition: there must be objective forms. The young man in white, who, in several MSS., speaks in the person of Jesus, was indeed he himself in his glorified being. Thus do I read the texts. Luke too had read something of this kind, which he reproduces thus:

But they were terrified and affrighted and supposed that they beheld a spirit. (Luke xxiv. 37. The Cambridge MS. and Marcion's edition of Luke both read "apparition" instead of "spirit.")

Let it be understood that I do not deny the possibility of ectoplastic phantoms, which Myers himself believed in, though he said he would not press them upon the credence of the reader, because of the difficulty of correct observation and the chances of fraud. Dr. Reichel of Germany has testified to their occurrence here in America. The difficulty in the New Testament is that they only appear in

the later accounts. Paul and (I shall show presently) Mark, our earliest witnesses, know of apparitions alone, not of materialized forms.

For the fullest account in English of all the problems the reader should consult *The Resurrection in the New Testament,* by Clayton R. Bowen, of Meadville, Pennsylvania (New York, 1911). Professor Bowen is one of a long series of laymen and liberals, like Griesbach, Lachmann, Tischendorf and Tregelles, who have taken the New Testament out of clerical hands. The three German lay professors and Tregelles, the English Quaker, were the ones whose work led directly to the Revised Version of 1881; but the task of revision is by no means ended yet.

Bowen was a Unitarian minister, but is now professor at Meadville. Before reading him, a shorter and clearer book by Kirsopp Lake should first be mastered.

* * *

Kirsopp Lake, of Harvard University, published in 1907 *The Historical Evidence for the Resurrection of Jesus Christ* (London, Williams and Norgate). Professor Lake at that time held the chair of New Testament Exegesis in the University of Leiden, to which Rendel Harris was elected in 1903, but did not serve. The book appeared in the Crown Theological Library and has been widely read. It contains a masterly analysis of the Resurrection narratives in 1 Corinthians, the Synoptical Gospels, the Acts of the Apostles, the Mark appendices, the Fourth Gospel and the apocryphal ones of Peter and the Hebrews. The conclusion reached is that Paul and Mark's accounts are historical, and the later ones exaggerated. Babylonian and other resurrection theories are reviewed, and the book ends with an allusion to Myers and psychical research. F. C. Burkitt, of Cambridge, in placing the essay in a

bibliography, says: "I introduce this book here as the first example in original English work of the doctrine of the priority of Mark being consistently applied throughout an historical investigation." (*The Earliest Sources for the Life of Jesus.* Boston, 1910, p. 129).

The method is that of the Lower Criticism, though the Higher is also freely used. What I especially wish to criticize is the following passage (pp. 61-65) which here we must read in full:

"*The young man at the tomb.*—The account of what the women saw at the tomb is contained in Mark xvi. 5. Dependent narratives are found in Matthew xxviii. 2-5 and in Luke xxiv. 3-5.

"*And entering into the tomb, they saw a young man sitting on the right side, clothed in a white garment; and they were astonished.*

"As it stands in Mark, this account gives rise at once to two questions: Did they see for themselves that the grave was empty? and who was the young man who appeared to them? Neither question is answered in Mark, but before considering the bearing of this fact, it is first necessary to ask whether the version given above represents the original text. According to it, the women entered the tomb and found a young man seated within on the right hand. No other meaning can be extracted from it, or ever could have been, in the presence of the word εἰσελθουσαι, 'entering into,' in verse 5 and the reference contained in the corresponding ἐξελθουσαι, 'going out,' in verse 8. But in case of neither of these words is the text perfectly certain. The former is in the Vatican MS. weakened to ἐλθουσαι, 'coming,' while the latter is not represented in the Arabic Diatessaron, and in some MSS. is altered to ἀκουσαντες, 'having heard.' The weight of textual evidence is against these alterations, but on the other hand transcriptional probability is in their favor. It is

unlikely that later scribes would have introduced changes in the text which were calculated to weaken the evidence for the belief that the women had made a complete examination of the tomb, and if these changes be made, the text of Mark would leave it doubtful whether the women saw the young man on the right hand of the inside or of the outside of the tomb; for ἐλθουσαι εἰς το μνημειον need not mean more than 'when they came to the tomb.' Is it possible that this represents the original form of the narrative? In the absence of other evidence, it may not be ill-advised to consider the evidence of a comparison with the two other gospels, Matthew and Luke, which are closely based on the Marcan narrative, and of the Fourth Gospel and the Gospel of Peter, which follow it with greater freedom. It has already been seen, in cases in which the Marcan document is undoubtedly ambiguous or difficult, that the dependent narratives adopted divergent methods of elucidating the points at issue. It may therefore be allowed to reverse this argument and see whether the dependent narratives in the present case support the suggestion that the ground document was ambiguous. They certainly seem to do so. Matthew represents the angel, who is in his narrative the equivalent of the young man of Mark, as seated on the stone which he had just rolled away; he was therefore regarded by Matthew as *outside* the tomb. It is equally plain that Luke regards the two men, who in his narrative represent the Marcan young man, as appearing *within* the tomb. Furthermore, the Fourth Gospel and the Gospel of Peter narrate that the women did not enter the tomb, but stooped down and saw an angel or angels sitting within. These two last accounts may quite well represent an attempt at conflation between two traditions which differed, or were not explicit, as to the position of the women and the angel with regard to the tomb, and so far they support the suggestion, which is rather strongly made by Matthew and

Luke, that the ground document was ambiguous on this point. The weak point in this argument is that it does not take account of the possibility that Matthew altered the Marcan document owing to the influence of the story of the watchers. It could be argued that the angel had to be kept in the presence of the watchers and of the women, and that the word ἀπελθουσαι, 'going from,' in verse 8 is a proof that the ground document of Matthew contained an account of an actual entry into the tomb. This is perhaps not a convincing argument, but it may be taken as practically balancing the previous one. It is impossible finally to decide between the two. I think that the balance of probability remains slightly in favor of the view that the original Marcan document narrated the story of the vision at the tomb in such a way, as not to state plainly that the women entered the tomb, but I should not be prepared to put emphasis on the argument."

I hope to show that there is every reason for Professor Lake to emphasize the argument that the original text of Mark did actually keep the women outside the tomb. We may say *does actually,* for the original text of Mark can be reconstructed from extant manuscripts and versions, without any appeal to the Higher Criticism. In one case only do we have to appeal to a lost source, but even this is supported by a patristic quotation, and therefore belongs to the Lower Criticism.

Let us begin with this lost source. Eusebius, in his *Questions of Marinus,* Question 1, which deals with the absence of the Mark Appendix (Mark xvi. 9-20) from the oldest manuscripts, says this:

"He who rejects the passage itself might say that the story does not exist in all the copies of the Gospel according to Mark; at least, the accurate ones among the copies describe the end of the story according to Mark in the words

of the youth who appears to the women, saying to them: *'Be not astonished; ye seek Jesus the Nazarene,'* and so forth. *'And when they heard, they fled, and said nothing to anyone, for they were afraid of....'* For herein the end is described in nearly all the copies of the Gospel according to Mark, and what follows is seldom found in any, but would not be superfluous in all, and especially if they should contain a contradiction to the witness of the rest of the Evangelists."

We may remark that "afraid of" is Kirsopp Lake's own translation of the concluding words of the genuine text of Mark, and it has been adopted by James Moffatt in his splendid translation of the New Testament (London, 1913). But the words for which we have copied this famous passage of Eusebius are: "when they heard" (ἀκουσασαι). Now it is known that Eusebius had access to the library collected by Origen in the third century and extended by Pamphilus. Indeed Conybeare has made use of this fact to delete the trinitarian formula and the baptismal charge at the end of the Gospel of Matthew, in the teeth of all existing manuscripts. He shows that Eusebius read Matthew xxviii. 18-20 without these theological additions, and places over against three thousand extant MSS., all later than the fourth century, that other thousand, now lost, which went back to the third and the second.

Applying this principle we can put in the forefront of our textual evidence for ἀκουσασαι instead of ἐξελθουσαι the whole weight of the earliest Christian manuscripts. The ungrammatical ἀκουσαντες quoted by Lake is from a medieval manuscript in Russia, numbered 565 by Caspar René Gregory in his Prolegomena to Tischendorf's Greek Testament. Of course Eusebius gives the right reading, ἀκουσασαι (feminine). Rallying to the support of this ancient Greek original are the Washington manuscript and the Old Syriac and Old Armenian versions, overlooked

UMI

by Lake. Their testimony is very important; especially the Armenian, for the Old Syriac and the Washington Greek betray a transition stage which was tautological. The "went out" was evidently interpolated before the deletion of the "having heard."

The following table will give a view of the process of corruption. As Eusebius expressly tells us that the most accurate MSS. omitted the Mark appendix, we need only deal with those that do so. This gives us a sure criterion. Six MSS. that omit this can therefore be pitted against 6000 that add it. To the trustworthy ones we may add those which contain the spurious matter with a caveat, also those which have a different ending from the current appendix. To these also must be added a few MSS. that contain attestations of careful copying from Jerusalem copies, such as No. 565.

Lost MSS. of the Early Centuries quoted by Eusebius.

[First clause not traced.]

And when they heard they fled and said nothing to any one, for they were afraid of.....

(*End of Mark.*)

Armenian Version.

And entering into the sepulcher.

* * *

And when they heard, they fled from the sepulcher, because they were terrified; and they said nothing to any one for they were afraid.

Gospel according to Mark.[1]

* * *

Introduction to Luke.

[1] The colophons here printed in bold-faced type are rubricated in the original.

Frank Normart, of Glenolden, Pennsylvania, but a native of Erzerum, has translated for me the passage from the Old Armenian, as found in his own printed edition (Constantinople, 1895) and in a valuable manuscript owned by John P. Peters (Bedrosian) of Philadelphia. (The colophon is from the manuscript, for the Bible Society has printed the Appendix, as in the King James version, with a note accusing the Greeks for omitting it, but carefully suppressing the fact that nearly all Armenian MSS. before A. D. 1100 omit it also.)

Both in the Syriac and the Armenian this colophon is rubricated.

Washington MS.

And entering into the sepulcher

* * *

And when they heard, they went out and fled from the sepulcher, for fear and astonishment had come upon them, and they said nothing to any one, for they were afraid of. Now when he was risen early, on the first day of the week, he appeared to Mary Magdalene etc.

[This is the earliest MS. that contains the Appendix, which it has in an unusual form, hitherto only partially known from a fragment in Jerome.]

Old Syriac.

And they entered into the sepulcher

* * *

And when they heard, they came forth and went away and said nothing to any one, for they had been afraid.

ENDETH GOSPEL OF MARK.

The South Coptic
(Sahidic or Thebaic.)

[First clause wanting.]

* * *

And when they had heard, they came out of the sepulcher, and they ran, for a trembling was laying hold on them, and a confusion; and they said not any word to any one, for they were fearing. But all the things which were ordered them, to those who followed Peter they said them openly. After these things also again Jesus was manifested to them from the place of rising of the sun unto the place of setting. He sent through them the preaching which is holy and incorruptible of the eternal salvation. Amen.

But these also belong to them.

[Then follows the Longer Appendix, after a repetition of the words at the juncture.]

The Vatican MS.

And coming unto the sepulcher

* * *

And they went out and fled from the sepulcher, for trembling and astonishment had come upon them; and they said nothing to any one, for they were afraid of.....

ACCORDING TO MARK.

The Sinaitic MS.

And entering into the sepulcher

* * *

And they went out and fled from the sepulcher, for trembling and astonishment had come upon them; and they said nothing to any one, for they were afraid of.....

Gospel according to Mark.

The Old Latin at Turin.

And when they had entered, they saw a youth etc.

* * *

But when they went out from the sepulcher, they fled; for trembling held them, and awe by reason of fear.

But all things whatsoever that were commanded, those also who were with the boy briefly explained. And after these things Jesu himself appeared, and from the East even unto the East (*sic*) he sent through them the holy and uncorrupted [preaching] of eternal salvation. Amen.

Endeth Gospel according to Mark. Beginneth happily according to Matthew.

The Ethiopic version also omits the Mark appendix, while medieval MSS. L, and Nos. 1 and 209 show the doubt about it; L gives both endings, like the South Coptic, putting the shorter appendix first as we have seen already. Nos. 1 and 209 say at xvi. 8:

"In some copies, the Evangelist ends here, as far as Eusebius the [friend] of Pamphilus, has placed his canons; in others, there are found also these [words]:

"NOW WHEN HE WAS RISEN," etc.

Several Fathers support the testimony of Eusebius, so that the proof is overwhelming.

By restoring "when they heard" at xvi. 8, we get rid of the ἐξελθουσαι, but this was introduced as correlative to εἰσελθουσαι: the two stand or fall together. If we had not a single manuscript that read ἐλθουσαι at xvi. 5, the Higher Criticism would bid us read it. But our very oldest Greek authority reads it, plus an eleventh-century MS numbered 127, together with the fourth-century Gothic

version. This reads *atgaggandeins,* "coming at," or "going unto." Lower Criticism therefore permits the restoration.

So we have unimpeachable ancient testimony that there was no ἐξελθουσαι at Mark xvi. 8: the women did not flee out of the tomb, because they had never been in it.

It is vain to protest that we cannot put three authorities against three thousand that read εἰσελθουσαι, "entering into"; for, by the laws of the Lower Criticism, we can, not only on the grounds already given, but by a well-known law of textual criticism. Dean Alford, in the critical apparatus to his Greek Testament, gives us the reason:

"Received text, εἰσελθουσαι, *from the parallel in Luke.*" (Henry Alford, *Greek Testament*: New York, 1859, Vol. I, p. 391.)

Nay, more: the Dean of Canterbury Cathedral did not hesitate to put ἐλθουσαι into his Greek text and to translate it in his *New Testament for English Readers* (London, 1868, Vol. I):

"And when they came to the sepulcher," etc.

The principle upon which Alford did this is perfectly sound. It is thus expressed by Jerome in his letter to Pope Damasus, introducing his novel Vulgate edition of the Gospels in the year 384:

"Great error, if indeed it be (so), has grown up in our codices, so long as what one Evangelist has said further on the same thing, (the scribes), have added in another because they thought it too little. Or, so long as another expressed otherwise the same meaning, he who had read first any one of the Four, considered that the rest ought also to be amended to the pattern of that one. Whence it happens that among us everything is mixed, and *there are found in Mark more things of Luke and of Matthew,* and again in Matthew more things of John and of Mark, and in others of the rest things which are peculiar to others."

Here we have the Protestant reason, stated by the

prince of Catholics, for modern revision of the text. Although we have not access to so many ancient manuscripts as Jerome had, yet we have ancient versions neglected by him, as well as two Greek codices of his own time and several Latin ones. We are more justified in ceasing to regard his work as final because he tells us himself that he did not do it thoroughly:

"This short preface offers only the four gospels, the order of which is as follows: Mathew, Mark, Luke, John, amended by a collation of the Greek codices, but (only) of old ones. Lest, however, they should differ much from the accustomed form of the Latin reading, we have so restrained the pen that, when such things only were corrected as seemed to change the sense, we suffered the rest to remain as they were."

We are now in a position to reconstruct the end of the Gospel of Mark, and to show that this most historic of all the Evangelists never told a story about a corpse that got up and walked off, but simply of some women who came to a tomb and saw a strange young man. When they saw him they were astonished, but when he addressed them they were terrified and ran away. At this point the Gospel ends, as we now have it. The reason for this abrupt ending requires a separate discussion, such as briefly outlined in our prologue or such as I attempted in *The Open Court* at Easter, 1910. Unfortunately I had not then read Kirsopp Lake, or my attempt would have been better. (At the top of page 133 I made a blunder; line 1 should read: "Now this note of doubt is Marcan, not Matthaean," etc.)

In giving the following text, several readings differing from the common ones have also been given in addition to those noted. Thus the phrase "on their right" in verse 5 is from the Sinai Syriac, and is all part of the reformed readings, for it no longer smacks of the inside of the grave where a youth or an angel was sitting on the right side of

the corpse. The word "daughter" is also from the Old Syriac. The Greek has a blank here, which our common translations supply by "mother." But a second-century version in the language which Jesus spoke may be presumed to go back to authentic tradition.

Our reconstruction is guided by the Lower Criticism alone. If we were to venture upon the dangerous ground of the Higher, I should strike out the words: "He is risen; he is not here." These are lacking in important MSS. of Luke, and as Luke used the first edition of Mark, he probably did not find them therein. But, as they appear in all extant MSS. of Mark, the principles of the Lower Criticism require that they should be retained. Higher Criticism would also query the historicity of the spices and ointment. Matthew says nothing about them, but tells us that the women came "to see the tomb," and had no errand inside. John expressly rules out the proceeding by an elaborate embalming before burial. If Luke's εἰσελθουσαι could be copied into Mark, as Alford following the statement of Jerome and the abundant witness of the manuscripts would have us admit, why should not Luke's ointment and spices also have found their way into Mark in very early times? But here again we are faithful to the Lower Chriticism and insert the spices and ointment.

Another point. When Dr. Lake says: "ἐλθουσαι εἰς το μνημειον need not mean more than 'when they came to the tomb,' " does he not understate the case? Can we not confidently say that they do not mean more? Thayer, in his lexicon, long since pointed out that εἰς το μνημειον in John xx, the parallel passage to the one in Mark, means simply "unto the tomb." In verse 1, the Revised Version renders the phrase: "unto the tomb"; in verse 3, "toward the tomb"; in verses 4 and 8 "to the tomb." So too in John xi. 31 and 38. In New Testament Greek, therefore, εἰς το μνημειον means "unto the tomb," and in order to introduce

UMI

UMI

the idea of entrance, Luke and the copyists of Mark had to alter ἐλθουσαι to εἰσελθουσαι. Then, having gotten the women into the tomb, they must be gotten out again; hence the correlative corruption of ἀκουσασαι ἐφυγον into ἐξελθουσαι ἐφυγον, which in the Sinai Syriac is even more tautological:

they went out and went.

This text is a conflation, for it has already given the original reading known to Eusebius in his Cæsarean manuscripts:

And when they heard.

In fact, a close study of the documents reveals the fact that the whole passage has been systematically tampered with.

Mark xvi entire, as in the Oldest Manuscripts.
Revised Text.

---

230 VIII And when the sabbath was past, Mary Magdalene and Mary the daughter of James, and Salome, brought spices, that they might come and anoint him.

231 I And very early in the morning, the first day of the week, they came unto the sepulcher at the rising of the sun. And they said among themselves: Who shall roll us away the sepulcher stone? (for it was exceeding great). And when they looked they saw that the stone was rolled away. And coming unto the sepulcher, they saw a young man sitting on their right, clothed in a white robe; and they were bewildered.

232 II And he saith unto them: Be not bewildered; ye seek Jesus the Nazarene, who was crucified. [He is risen; he is not here.] Behold, there is his place where they have laid him. But go your way, tell his dis-

ciples and Peter: I am going to Galilee ahead of you. There shall ye see me, as I have said unto you.

233 And when they heard, they fled, and said nothing
II to any one, for they were afraid of.....

HERE ENDETH THE GOSPEL OF MARK.

The numbered paragraphs are known as the Ammonian Sections and appear in the Sinaitic manuscript and down through the Middle Ages. Underneath the Hindu numeral is the canon ascribed to Eusebius, which is numbered in Roman. These canons represent an ancient Gospel analysis of no mean ability: Canon I means that the section is common to all four Gospels; II, to the three Synoptists; VIII, to Mark and Luke; X, peculiar to one; the rest to different pairs.

RECAPITULATION.

From early manuscripts, versions and Fathers we can reconstruct the text of the Resurrection in Mark, so as to read, at xvi. 5, "coming unto the sepulcher," instead of "entering into" it. Comparison of these authorities reveals the fact that the text has been systematically corrupted in early times by altering "coming unto" to "entering into," and "when they heard," in verse 8, to "going out of." This has been done so as to make it appear that the women found the grave empty, implying the doctrine of a fleshly resurrection. By the Lower Criticism alone we are enabled to correct these corruptions and to show that Mark originally contained no account of a physical resurrection. The familiar problem of the lost ending of Mark is incidentally dealt with.

ALBERT J. EDMUNDS.

PHILADELPHIA, PA.

BERGSONISM IN ENGLAND.

M. BERGSON has told us that on the arena of Europe to-day we have a spectacle of "life" in arms against "matter." He takes the two terms to express respectively the spirit of his own nation and that of the enemy. We may take them as expressing at least the alternatives between which his own philosophy moves. On the arena of the universe as a whole life and matter are in conflict and his philosophy seeks to decide between the two. Which is the ultimate reality? Which is to be explained by the other? Is life a product of matter or is the truth the other way, has matter itself been created by life? Bergson's works may be considered to constitute an elaborate answer to this question, and to decide it in favor of the latter alternative. He thus takes rank as a champion of the living against the dead. But the main position against which he argues—what one could call roughly scientific naturalism —is one with which philosophy in this country has long been accustomed to deal, and with a like result. Now that Bergson's philosophy may be said to have struck its first roots in this country, the time seems opportune for a comparison. We might with profit compare what he has to say with what the philosophy of the past two generations has been teaching. From some points of view it seems as though his special followers did not realize sufficiently the likeness of the two ways of thinking. Nor do they seem to have seen as yet how much some of Bergson's most

valued positions seem to invite and indeed to demand strengthening by reference to matters which have been developed at least partially by the older idealism, and which are absent from him.

The opportunity for making the comparison is ready to our hand. As is well known, the head and front of Bergson's English following has come to be pretty much identified with Dr. H. Wildon Carr. While his work, *The Philosophy of Change*,[1] shows the influence of Bergson at every turn, and while Dr. Carr does not conceal his conviction that Bergson's is the most successful attempt yet made to deal with the questions of metaphysics, still it would be wrong to say that in this work he has merely been expounding another man's thought. He makes the endeavor to restate Bergson's central principle and to substantiate it by fresh applications; and a glance at the main topics will serve, we believe, to give a fairly true view of one important feature of Bergson's thinking which it is necessary to take account of, if we would bring its relation to English thought properly into focus. It will show us the *kind of questions* which figure in Bergson's pages.

The recurring topic is matter and spirit and the problems which arise out of their relation. For instance, is mind produced by the brain? If so, how could mind ever get at what is outside of this brain? for apparently mind can reach not only what is outside of it but what is even separated from it by immeasurable gulfs of space and time. Or, what are we to make of the question as to the constitution of matter, now that recent discoveries seem to resolve the material atom into something that is not a substance at all but only so much electricity? Then there are the problems connected with the relation of conscious mind to movement. Why is it that a higher form of consciousness, in the animal world, is so constantly accompanied by

[1] Macmillan & Co., 1914.

an increased capability on the animal's part of choosing its movements? There is further the nature of life itself. How are we to construe the fact that the whole past of every living being appears to be recorded in its present structure? And again there is the law of evolution and its ways of working. How is it that in a material world supposed to be governed by mechanical law, evolution can bend and govern the most dissimilar series of conditions so as to produce a like result—as when (to cull an illustration from Bergson himself) the series of conditions which produces a mollusk and the very different series which produces a vertebrate animal should both alike end in the one result, the endowment of the creature with an eye?

An outstanding characteristic of the whole way of thought will be apparent from this cursory survey. Its leading questions are such as would arise in the course of the study of natural science. They are questions which would occur to a scientist with philosophical interests. No doubt the motive which impels the mind to raise them lies far back in the perennial human sources of the philosophizing habit. But the actual questions raised come straight out of modern physics, mathematics, biology and psychology. They don't arise out of a study of the history of philosophical disputes. This is characteristic of Bergson. He is preeminently one of the writers who attack problems, not other people's solutions of problems. Hence his philosophical freshness. He does indeed deal with the history of thought. He deals also with some of the standing controversies of current philosophy. But these are not the center of his interest. He does not begin with these. His speculation thus acquires an interest for scientific minds which most philosophy does not possess. And it is perhaps not altogether fanciful to say that this feature gives his thought from the outset a certain advantage with the English mind. Moreover, in Dr. Carr Bergson seems to have

UMI

found a follower whose interests also are preeminently in the concrete present. The central matter with him seems to be, not what can we make of the systems of the past, but rather what can we make of the report which the sciences give us now of the world we are in; what is important is that report and the right interpretation of it.

Fortunately, however, what we have just referred to as the perennial sources of all philosophy—the great hypotheses of our emotional nature—are touched upon by Dr. Carr much less lightly and less fleetingly than by his master. He even makes bold to adopt as a heading for one of his later chapters the well-known words in which Kant summed up the whole demands of our higher emotional nature, the phrase "God, Freedom and Immortality." It is a fortunate circumstance for the comparison which we have in hand. In Kant we have the original source of that general idealistic view in philosophy which has practically held the field in English teaching until recently and with which we wish to compare Bergson's. These postulates will therefore give us a starting-point. We can compare what light upon them has been derived from the older sources, with what has been given to us by Bergson. And we can compare the two philosophies further as regards their ability to justify what they have to say on these things, and give a reason for the hope that is in them.

Before we can consider what light Bergson throws upon ultimate questions, or compare it with any derived from elsewhere, we must first try to gain some rough idea of his general view. It is clearly impossible to go into detail. We cannot indicate Bergson's actual answers to the questions we have cited above, still less his answers to all the questions of which these are only a few taken at random. But it may be possible to indicate his principle. It may be possible to point out what in the universe Berg-

son especially sees and values; what it is which he believes to be capable of providing a solution to these and all the various problems with which he deals. It is in point of fact nothing else than that *élan vital* which has figured so often in the reviews of the new French philosophy, that vital impulse which we behold forcing itself along the whole course of evolution and of which in the long run Bergson holds the universe itself to be the creation.

With this mere hint of the view, let us turn at once to the question how it stands to the idealism which has been taught in England and which finds its classical English expression in the writings of T. H. Green,—how it stands, especially, as regards an attitude to problems which are of the last importance for the human mind.

In the first place there are striking general resemblances. In Green's doctrine as in Bergson's the fundamental reality of the universe is not matter but consciousness. Like the former view, Bergsonism professes nevertheless to be neither "idealism" nor "realism" *simpliciter*. And up to a point it adduces the same reason for repudiating the former of these titles—for refusing, i. e., to identify its teaching with any such "idealism" as is usually associated with the name of Berkeley. The reason in both cases is this. The reality of the world, though for both views it is consciousness, is not for either of them anything constituted simply by our private minds. No thesis is put forward by either theory to the effect that you and I and other minds like us are all that exist. The consciousness referred to is in the literal sense universal. It is over all the universe, a feature of the whole of it. No such doctrine is put forward from either quarter as that the universe which we usually see and know does not exist. What is said is that it is conscious and is the product of its consciousness. But, for Bergsonism, that self-creative consciousness which the universe is differs from our private

minds also in another way, a way reminiscent of Schopenhauer as well as of the neo-Kantian idealism of Green. That consciousness is not preeminently representative or pictorial. It is active. Not the static picture of knowledge is its characteristic expression, but the energy of will.

In consciousness so conceived, then, Bergsonism finds the key to the broad facts of life and evolution as science has revealed them to us. In the evolutionary history of life on this planet—in the genesis and progress of vegetable world, animal kingdom and man—what we have is this active consciousness in the form of life, pushing itself, as it were, through the surface of matter and seeking free way. Man's physical organism is the one configuration of matter through which it finds the free course which it seeks. The human body is organized for giving outlet to this activity. The brain and nervous system are but its cutting edge by means of which it thrusts itself forward. The story of evolution is the story of how the main current of this vital impulse has worked its devious way through matter. The different forms of life which we see are the different channels into which the central stream has split itself up in process of thrusting itself into matter. The central stream has not quite dissipated itself as yet into these branches. The main current is still traceable. It is found in the life and consciousness of man. It is for this reason that man is at the head of creation. His life and mind contain the most complete concentration to be found anywhere of what was in the original world-impulse. The fundamental reality of the universe, then, is life; but it is a life which comes to view best, not in the plant or the animal but in the *conscious* life of man. We must note further that the "matter" through which the stream of life thrusts itself is in the last resort its own creation, though we need not go into Bergson's proof of that here. The vital impulse is thus creative of matter and of all the forms

of life in which it finds an outlet, and the whole process of its advance is named by Bergson "creative evolution."

Some such view is the only one capable, in Bergson's opinion, of meeting the necessities of the case which natural science has presented to us. Dr. Carr has endeavored to go further and show that as a general view it is specially in harmony with some quite recent scientific discoveries. The "vital impulse" is nothing if it is not movement. It is, in fact, pure movement. If it be creative of "things" then somehow things must be generated out of movement. And this, Dr. Carr points out, is precisely what science is now finding.

"The essential principle of the philosophy of change," he says, "is that movement is original. Things are derived from movement, and movement is not a quality or character that things have added to themselves."[2] "A very few years ago," he says again, "such a doctrine would have sounded paradoxical and absurd. But now compare the philosophical doctrine of original movement with the new theories of science. Let us take first the structure of the atom. The electrical theory of matter teaches that the atom is composed of a central mass or core, which is far the larger part of its substance, and an envelope small in comparison. The central core is positive electricity, and the outer envelope consists of negatively electrified particles held in position by the electrical relation to the central core. The atom, in fact, is a solar system in which the positive element is the sun and the negative element the planets. And all the qualities of atoms depend upon the arrangement of these outer negative elements. But what is the ultimate reality of this atom—something or other that is electrified? No, it is electricity, not something electrified, and electricity is a form of energy, and energy degrades and disperses.

[2] *Philosophy of Change*, p. 11.

Reduced to simple everyday concepts it is this, that what we call matter is a form of movement."[3]

But it is not merely in the case of the atom that recent discoveries have tended to resolve into terms of movement what we had been accustomed to regard under quite other terms; elsewhere also they have begun to transform the static into the changing, the resting into the active.

"But now turn to the other side," Dr. Carr continues (pp. 17-18). "In the last few years it has been possible to demonstrate that our solar system is not, as was supposed, at rest in an absolute space or else moving, if it be moving, without regard to forces outside itself. It belongs to a larger system, all the parts of which are in movement in relation to one another. The fifty million stars that our telescopes reveal are not scattered at random over the firmament, but are moving along regular courses coordinated to one another. The members of this stellar system are not, like the planets, revolving round a central mass, but millions of suns are streaming across an unoccupied center. The speed of our sun (now about 12½ miles a second) has been calculated, and its direction and the acceleration it will undergo as it travels across the center and passes outward again to the periphery. This, however, is not all. A discovery has been announced that seems likely to extend indefinitely further than astronomers have yet imagined the vastness of the spatial universe. Observations which have been made on the great spiral nebula in Andromeda show that its spectrum is inconsistent with the hitherto generally held supposition that it consists of gaseous matter in a state of extreme tenuity. It is now said to be a spectrum that is given out by solid glowing masses, and thus seems to confirm an old view that the nebulae are star groups immensely distant. This nebula is apparently not within our stellar system, but itself a vast stellar system

[3] *Ibid.*, pp. 16-17.

lying outside the latter and at an enormous distance away from it. What other systems lie outside these we do not know, but all that we discover suggests universal movement. There is no absolute rest. If we conceive an observer placed anywhere in this great universe that we look out upon from our position on an insignificant planet of an insignificant sun, whether we suppose him to gather into one embrace what to us are vast stellar systems or to be confined to the negatively charged ion of an hydrogen atom, there will stretch out for him on either side an unlimited expanse of reality of which the ultimate essence is movement."

And Dr. Carr finds a suggestion of the same point, viz., that things are not more original than movement but that movement is more original than things, in the way in which the recent "principle of relativity" in physics threatens to transform our conceptions of space and time and remove the ether from its place as a scientific hypothesis. All this seems to him to confirm the view expressed by Bergson in *La preception de changement*: "Movement is the reality itself, and what we call rest (*immobilité*) is a certain state of things identical with or analogous to that which is produced when two trains are moving with the same velocity in the same direction on parallel rails; each train appears then to be stationary to the travelers seated in the other." And again: "There are changes, but there are not things that change; change does not need a support. There are movements, but there are not necessarily constant objects which are moved; movement does not imply something that is movable."

The discovery that the whole universe is movement is, however, very little, if we know nothing about this movement except simply that it moves. Even when we have brought it so far that we can regard this movement as life,

UMI

as creative, and as able at last to burst into consciousness, we have not even then got very far philosophically unless we learn something of the inner character of this vast spiritual force. On its inner character too must rest our judgment upon Dr. Carr's bold claim for the "new method," that it is nothing less than a revolution and that it has reversed the direction that philosophy has followed throughout its history of 2500 years.[4] It is on the subject of the inner character of this movement that Bergson's teaching most directly challenges comparison with that of Green; and, we may add, most clearly demands to be supplemented by it.

For Bergson is by no means the only teacher who has conceived of a universal spiritual energy as sustaining the universe. Green teaches the same. In the words of by far the ablest existing short exposition of it, the central conception of his philosophy "is that the universe is a single, eternal, activity or energy of which it is the essence to be self-conscious."[5] Nor can we get a distinction for Bergson out of his repeated claim that the spiritual activity of which he is thinking is not purely conceptual, because Green in essentials makes the same claim. A great deal is made by Bergson of the non-conceptual character of true philosophical apprehension. You cannot apprehend that ultimate essence or spiritual force whereby the universe exists, in the ordinary way; that is, by the intellect. The reason why, is that the intellect can only apprehend what is dead, static, given. It cannot grasp living movement. Now Green has quite as little use as Bergson has for what can only grasp the given or static. Natural science, for Bergson, is based on the intellect and that is why it cannot conduct us into the presence of what verily explains things.

[4] See *Philosophy of Change*, p. 20.

[5] R. L. Nettleship in his biography of Green: Green's *Works*, Vol. III, p. lxxv.

Science only sees in the universe what is dead, and therefore it cannot exhibit its ultimate spiritual essence. This is Green's complaint too. Green, indeed, does not say that we must appeal to something else than the intellect (this is Bergson's way of putting the matter), but he does say that we must understand the intellect. We must not be content to use it uncritically, as natural science and naive common sense do. We must lay hold of life and activity with it. But Green is clear that the life and activity of which the intellect must lay hold, is its own. His philosophic creed, shortly stated, is that this is possible—that intellect, itself an energy, will reveal a spiritual energy at the heart of the universe, if it be persevered with and rightly used. Green does not say so in these words. But his philosophy says so to any one who has entered into it. He insists on the one hand that reason and will are one, in the sense that they are alike expressions of one principle;[6] and he speaks constantly of that principle as active and as self-active. His phrase is "self-realizing." On the other hand, his whole contention against the empirical school in philosophy was that this self-activity, the essence of men's minds, was not in men's minds alone. It was the essence of the universe. The spiritual principle was "in knowledge," and also it was "in nature," as the most elementary student of his chief work soon learns to know. Practically all he had to say about nature in fact was just this: that it was not inert, dead, merely given; that it was a spiritual life, of which our individual minds were the highest finite manifestation. So far Green and Bergson are on common grounds.

There is, however, a real divergence between Bergson's and the older teaching. They differ in their doctrine of time. Both agree that what we can see around us with

[6] See, *inter alia*, *Works*, II, p. 329.

the bodily eye is not the ultimate spiritual energy but its manifestations only. Even with the eye of the mind, they both hold, the ultimate spirit itself cannot be apprehended in its whole nature, for only part of its original totality is contained in the mind of man. But they differ as to what it is in its own whole nature. With Green it is in itself already perfect, whereas with Bergson it is still developing and changing with the course of time and has an immense and entirely uncertain future ahead of it. With Green the ultimate spirit is complete, with Bergson it is incomplete. With Green it is a consciousness, morally and intellectually all that we could conceive ourselves becoming. With Bergson it is a consciousness still always turning into something different and turning always into something which could not have been predicted.

At this point also occurs the most marked difference of the two doctrines in regard to the light which they cast upon the assumptions of the moral and religious consciousness. And as regards religion, it is not hard to judge which is in the stronger position. So far as the religious mind has entertained the belief that behind the phenomena of the universe and acting as their source, there exists a mind which is eternal, one who is above time and vicissitude, who is perfect and is not subjected to change, a God "who was and is and ever will be," in so far it will find its faith countenanced in Green's teaching but discountenanced in Bergson's. Dr. Carr himself is clear that the change Bergson's theory invites us to make in our religious conceptions is profound, though he thinks that it has compensations.

"How is the conception of God affected by the principle of this new philosophy? One attribute that has seemed to attach to this conception can certainly not belong to it—eternity, in the sense of timelessness. Reality is essentially movement, movement is duration, duration is change. If

we call the original impulse of life God, then God is not a unity that merely resumes in itself the multiplicity of time existence, a unity that sums up the given. God has nothing of the already made. He is not perfect in the sense that He is eternally complete, that He endures without changing. He is unceasing life, action, freedom.

"No more profound change can be imagined in the conception of the universe, in the conception of human nature, in the whole outlook of life, than is involved in this new conception of God. The conception of God to which we have been accustomed in philosophy,—the most perfect being, the *ens realissimum,* the first cause, the *causa sui,* the end or final cause,—is the conception of a reality which time does not affect. Hence the continual attempt both in ancient and modern philosophy to conceive two orders or kinds of existence, the temporal and the eternal, and the whole problem of philosophy has been to conceive the relation of these two orders to one another. Time and the whole order of changing reality must, it has seemed, be of the nature of an emanation from God, or a manifestation of God. But however conceived, the time order is regarded as essentially unreal, appearance and not reality; change and movement are relative to us."[7]

Connected with the same difference in regard to time, there is again a difference of the two theories as regards the light they throw upon another of the Kantian postulates. So far as the religious consciousness has fixed its hope on immortality in the sense of a life out of time Bergsonism can offer no corroboration, for time and the change which constitutes it are to this philosophy reality itself, and to be out of time is *ipso facto* to be out of existence. Here again the older view is considerably different. For it the idea of a life beyond time is at least not contradictory. Nay, any completing or perfecting of our best life here

[7] *Philosophy of Change,* pp. 187-188.

would inevitably have this character, since for this view we are already above time in so far as we think what is true and do what is unselfish.

As regards the postulates of God and immortality, then, the effect of Bergsonism is of a negative character. But these are preeminently religious postulates. The point upon which Bergsonism claims most confidently to have substantiated our higher emotional demands is in regard to the moral postulate, that of freedom. In its clearness upon this question, indeed, Dr. Carr finds the chief compensation for its attitude upon the others.

"The philosophy of change does not sound any clear and confident note as to what lies beyond us in the unseen world. It does not present to us God as the loving father of the human race, whom He has begotten or created that intelligent beings may recognize Him and find happiness in communion with Him. There may be truth in this ideal, but it is no part of philosophy. Neither does it teach us the brotherhood of the human race—on the contrary it seems to insist that strife and conflict are the essential conditions of activity. Life is a struggle, and the opposing elements are the nature of life itself, the very principle of it. The evolution of life is the making explicit of what lies implicit in the original impulse. Philosophy reveals no ground for the belief in personal survival, and it shows us that however highly we prize our individuality we are the realization of the life-impulse which in producing us has produced also myriad other forms. What then is the attraction that this philosophy exercises? What is there of supreme value that it assures to us? The answer is freedom."[8]

Here at length we reach the philosophically important matter. For here we can interrogate the two views, not merely as to whether they can corroborate our religious

[8] *Philosophy of Change*, pp. 195-186.

sense, but as to their grounds for doing so. The whole question for the critical evaluation of the philosophy of Bergson, it may be said, is that of the nature of and the evidence for the freedom which he says characterizes that ultimate spiritual force of which we are the offspring, and which by its vast uprush through the universe and through us creates us and the universe as it goes.

For Green too, as everybody knows, there is freedom. And he puts the *rationale* of it thus. Man is free, for him, both in his knowing and his acting, because in both of these functions the past is gathered up in the present which is now before him. Except this were so, says Green, we could not know. To know, is to know succession. Now if there were only succession itself—that is, if the past were not thus gathered up—there could be no consciousness thereof. This is straightforward reasoning, and at bottom quite simple. If I am gathering a bunch of flowers, I must hold the first ones in my hand while I gather the rest. If I did not do this but dropped each one as I picked it, I should never have a bunch. Quite similarly, if I hear or see a succession, say, of strokes upon a knocker, and if I *know* that it is a succession of five knocks, my knowing is evidence sufficient that the earlier strokes have not escaped me but have been gathered up in my mind and presented along with the last one. If each had disappeared as it occurred there would have been no succession of five for me. Each one would have been number one; and when it was over would have been nothing. To perceive time at all I must not merely have the present before me. I must have the past along with the present. In Green's phrase, the various members of the series must be "co-present" to consciousness.

Bergson has made an analysis of this same experience, and has given the matter profound attention. He too sees, that to know succession in the ordinary sense of knowl-

edge the members of the succession must be somehow co-present, but he gives the whole matter another turn. He cannot feel, apparently, that in knowing the successive as thus co-present we are really knowing the successive at all. His refrain therefore is, we *try* to know a time-succession by the ordinary use of our intellect, but cannot. We do not, in this fashion, know a time-succession. We only succeed in knowing space. In counting the strokes we set out the series of events in a row, along a line, in a kind of mental space. This we call perceiving their temporal succession. And if one asks, "Why does the intellect fail? how are we to apprehend time, or what would it be like if we would apprehend it?" the whole argument of Bergson's *Time and Free Will* converges in effect upon this answer: that the intellect which fails to apprehend time-succession fails because it can only set out the events *separately* along an imaginary spatial line, whereas for the "intuition" which really apprehends time these events are not separate, they interpenetrate. This interpenetration *is* time. It is fairly easy to see further how, out of the apprehension of such time, he gets free will. We have to pull ourselves together in order to grasp this interpenetration; and in this attitude, in this tense summoning of ourselves together, we are free.

We have here the fundamental impeachment of reason to which Bergson's philosophy seems compelled to have recourse. To reject the intellect as a means of attaining to the truth is an obvious weakness, as compared with the other view, thus far—that it is a species of self-subversion which the view with which we are contrasting it does not commit. Both Bergson and Green in philosophizing at all are endeavoring to settle their account with the problems of life by thinking them out. Both, in other words, are making use of the intellect. The difference between them in regard to the matter before us is that Green trusts the

instrument he is using. Having found what the intellect perceives time and succession to be, he says frankly that that is what they are. But Bergson, unable to accept the verdict, will rather make bold to say that our rational mind is incompetent, that it is incapable of seeing things as they are, and so has no authority in the case. This is a serious matter. One cannot feel, after this, that the intellect can be a very safe instrument to philosophize with. This is perhaps the rock on which all philosophies eventually split which attempt to reason the reader into preferring some supra-rational or sub-rational power before reason itself. Mere reason may not be fit to see what reality is; but if not, is it fit to attack itself either? We cannot endorse this intellectual abuse of the intellect. If the intellect cannot justify itself it cannot justify anything. We must accept the intellect, or our whole attitude is sceptical.

"But the intellect can't allow you free-will," it will be at once objected. This is an ancient objection, of which, as we shall see, Bergson himself shows us how to get the better. What, we have to ask,—what precisely is the freedom that Bergson's argument itself will bring us if it is true? It is easy for Dr. Carr to speak as if Bergson preserved for us the privilege of a wide choice in an open universe. All defenders of freedom have used such language. The question is, what evidence has he? What is there in our own experience that we can fall back upon and see that the universe is open before us? What reveals our identity with a universal principle of freedom which creates the universe itself, and in whose life we are free?

Whatever answer can be got out of Bergson to the question must come from the "interpenetration" just mentioned. And on inquiry we find that it is a solid answer enough. We do get evidence of freedom. And it is from the "interpenetration" that we get it. Bergson is one of the few people who see where the freedom issue really lies.

UMI

In *Time and Free Will* he insists that freedom is to be looked for in the character of an act itself. It is the question "what was the act?" that is essential; not the question "what might it have been?" or "could it have been different?" What we have to ask about two alternative courses of conduct ahead of us, when we want to know whether we are free agents, is not "is either equally possible to me now?" but "what is the inner character of the one chosen when it does eventuate?" And he indicates, in language which might have been copied from Green, that our character must be in our act. "We are free," he says in *Time and Free Will,*[9] "when our acts spring from our whole personality, when they express it, when they have that indefinable resemblance to it which one sometimes finds between the artist and his work. It is no use asserting that we are then yielding to the all-powerful influence of our character. Our character is still ourselves;" etc., etc. And what we learn from his lengthy subsequent discussion of the matter is simply this: that where "interpenetration" occurs, there our character is; where the multiplicity consciously present in us is made up of items which interpenetrate, there our personality has its seat. And where the multiplicity of interpenetrating states is at its maximum in the great, critical decisions of our life, there our freedom is at its maximum because our personality is so. "It is the whole soul....which gives rise to the free decision; and the act will be so much the freer, the more the dynamic series with which it is connected tends to be the fundamental self."[10]

It takes a great effort, often, to draw the scattered multiplicity of our conscious states into this interpenetrating unity. And in his later work, *Creative Evolution,* Bergson tries to show that when this concentration of spirit

[9] English Translation, p. 172.
[10] *Time and Free Will,* p. 167.

is relaxed an order of freedom transforms itself into an external order of necessity. There is no disorder in spirit, but only these two opposite kinds of order. That is how he accounts for matter. It is the de-tension of the universal life-impulse. But the present point is, that an act is free when our personality is in it, and that happens when it is one such as gives outlet or utterance to a multiplicity of states held in an intense interpenetrating unity.

So far, Bergson conducts us along safe and solid ground. But let us not make a mystery of this interpenetration. The highest examples of it are to be found only rarely, no doubt. We find them in moments when the entire being of a richly endowed mind, all its desires, fears, hopes, knowledge, emotions, converge in one direction, meditate one high and hard decision, and that decision is taken. There you have that contracting together of the entire soul for the effort, of which Bergson speaks under so many similes, and which is perhaps the highest act of a life. But there are simpler examples. The simplest is the common experience we have already referred to—the mere watching a series of events go by. The vague impression left by the last "click" of a series to which we have not been attending will tell us, says Bergson, (if we start up afterwards and try to count how many we have missed) when we have counted enough. In such a case the objects we consciously count are set out in a sort of mental row. Not so the vague impression which acts as our standard and says to us when we have counted up, say, four, "that is enough." This vague impression does itself contain four. It is an impression of four. But it contains them in a different way. In it they are not set out in a row, but interpenetrate. Its "four" character, its quadruplicity if you will, is a unique quality.

"Whilst I am writing these lines, the hour begins to strike upon a neighboring clock, but my inattentive ear

does not perceive it until several strokes have made themselves heard. Hence I have not counted them. Yet I only have to turn my attention backwards to count up the four strokes which have already sounded and add them to those which I hear. If, then, I question myself carefully on what has just taken place, I perceive that the first four sounds had struck my ear and even affected my consciousness, but that the sensations produced by each one of them, instead of being set side by side, had melted into one another in such a way as to give the whole a peculiar quality, to make a kind of musical phrase out of it. In order, then, to estimate retrospectively the number of strokes sounded, I tried to reconstruct this phrase in thought: my imagination made one stroke, then two, then three, and so long as it did not reach the exact number four, my feeling, when consulted, answered that the total effect was qualitatively different. It had thus ascertained in its own way the succession of four strokes, but quite otherwise than by a process of addition, and without bringing in the image of a juxtaposition of distinct terms. In a word, the number of strokes was perceived as a quality and not as a quantity; it is thus that duration is presented to immediate consciousness, and it retains this form so long as it does not give place to a symbolical representation derived from extensity."[11]

Now the freedom which Bergson secures, and which he says cannot be apprehended by the intellect but only by what he calls "intuition," is this interpenetration. The intellect, he holds, cannot grasp it. But if we put aside his statement that the intellect cannot grasp this unity of interpenetrating items, and attend solely to his description of what the intellect is alleged not to be able to grasp, we find that his statement is quite wrong. The intellect can grasp it, and Green's doctrine is precisely that it can. True, "interpenetration" is not a favorite word of Green's. He

[11] *Time and Free Will,* Eng. Trans., pp. 127-128.

speaks of relation. He holds that the members of a succession in order to be known to our minds as a succession must be related; so related that they are co-present. But this interrelation which Bergson says is a misreading of time and a translation of it into mere "space symbolism" because the members don't interpenetrate, this intellectual apprehension of a succession, is already to Green precisely a complex of interpenetrating elements. True, the items are connected by relation, but relations are internal for Green. They are constitutive of the thing's character. The relations in which each thing stands to the others are what make its nature. The nature of all the others, therefore, enters into each, and that of each into all the others. They must interpenetrate; their natures do so as truly and literally as two brushes which have been stuck together. The fact is, it is altogether the same whether we say of certain elements that their mutual relations are internal to each of them, or that they penetrate one another.

"But this is not the interpretation that Bergson means," it will be replied at once. "This interrelation of Green's would never yield anything like freedom. What Bergson means is a vital interpenetration, not any dead static thing such as could be illustrated by the mere material interpenetration of the bristles of two brushes." Entirely so. The metaphor does not do justice to Bergson's position, and neither does it to Green's. With Bergson the interpenetration of the elements seen by "intuition" is vital, it is an intense living movement, and he strains language to express how the elements fuse together, melt into each other, inter-work and support a real life. But neither, with Green, are the objects of *the intellect* in a dead *relation*. A relation, with him, is a relating—a living activity, therefore. He has nothing to teach if he does not teach this. He has nothing to urge if he does not urge that a system of relations "implies a relating mind." And surely

no one ever took him to mean by that, that the implied "mind" merely made the system, set it down, and left it for ever alone to stand there, dead, cold and finished. The relations are alive. They are being kept up. They are a deed; and not a deed done but a deed ever a-doing. A relation of two things, with Green, is a supporting of them in an energy of ceaseless spiritual movement, in precisely the Bergsonian sense.

"But this movement constitutes the things, with Bergson; it is their source, the very stuff of which they are made." Even so with Green, and much he has been made to suffer for it! It is not, says Bergson, things which are first and which come to interpenetrate afterward. It is the movement or interpenetration which constitutes the things. It is not, says Green, things which are first and which come to be related or interpenetrated afterward. It is their relation or interpenetration which constitutes them. A thing is nothing apart from its relations.

So far as regards the tracing of reality to a spiritual source Bergson indeed uses a language which is different from that of the older idealists. But in this general matter his fundamental thought is accurately the same. The only difference is that the older teaching does not fall back on any special intuition in order to be assured that reality has a spiritual source. It relies on the more thorough application and the critical use of the intellect itself. It holds that this most important of truths still is truth, and that by those who persevere it may be reached by the same methods through which other convincing truth is reached, namely, by the exercise of reason.

But this one difference is a difference as of heaven and earth. By disparaging intellect it puts Bergson in the unhappy position of constantly needing to discredit that very faculty of "reasoning" upon which as a philosopher he

must stake his own results; and that is not the whole of the trouble. It also gives a false cast on the moral side to the entire physiognomy of his teaching. And with a glance at this we may close our review.

The significant point is that Bergson does not believe in the intellect, or in the typical object of the intellect, namely, space. By not believing in them we mean that he does not believe in their spirituality. Green does. Green finds in space itself that very interpenetration or spiritual movement which Bergson insists cannot be found there. He finds, that is to say, in the (spatial) object of the intellect something which fully answers the essentials of Bergson's description of the interpenetrating, while Bergson constantly speaks of this character in things as though it could not be seen at all intellectually, but only in glimpses, by the special power of apprehension which he calls intuition. Green, in a word, finds in the spatial-intellectual that reality and truth which Bergson can only find when all "space-symbolism" has been done away with. This is a serious difference. For this "space-symbolism," in the wide meaning which Bergson gives to it, is the very stuff and fiber of the moral life. His teaching therefore means that to be at the moral point of view is to be out of touch with the real truth of the world.

And unfortunately his actual ethical teaching bears out the suggestion. It is quite a mistake, we may note in passing, to say that Bergson has not written on ethics. It is true he has not written any book with that name. But he has a work the real burden of which is an interpretation of the moral and social life. This is his little treatise *On Laughter*. His thesis in that work is that laughter is a species of social castigation. It is designed to rid society of the conduct that provokes it. And the question for the moral implications of Bergson's teaching is, what is it whose destiny is thus to be socially castigated? Startling as the

answer may seem, it is the moral. It is called the mechanical. In the wide sense in which Bergson eventually uses the term, it is the intellectual-spatial. But in the concrete what is it? It is simply faithfulness to principle where such faithfulness is awkward. In other words it is the very soul of the moral life, if that is anything at all distinct from the "esthetic" life. This disbelief in space and the spatial, this disbelief in the negation which is at the root of these, is what the present writer has ventured to call the pessimism of Bergson.[12]

Without repeating here what has been worked out elsewhere,[13] reference may be permitted to one little point in elucidation of this view. It concerns Bergson's first illustration in *Laughter,* his picture of the runner who stumbles and falls. It is a small matter, of course, but it has always struck the present writer as a peculiarly significant accident that Bergson should have opened an essay *On Laughter* by taking as his first example of the ridiculous precisely that figure which has served so many moralists for their type of the moral life. The runner of Bergson's illustration, as Bergson describes him, with his eagerness and his "rigidness," with his omitting to look where he is going, his stumbling over obstacles and his abundant inability to adapt his conduct as circumstances require, and follow the sinuosities of his crooked path, is indeed ridiculous. But it is only Bergson's light vein that makes him so. There is nothing essentially ludicrous about such a man. In essentials, he might be Bunyan's pilgrim fleeing toward the wicket-gate or St. Paul's runner, who also heeds nothing

[12] See articles in *The Hibbert Journal* for October 1912, *The International Journal of Ethics* for January 1914 and *Mind* for July 1913. Compare an article on "Bergson, Pragmatism and Schopenhauer" by Günther Jacobi in *The Monist,* Vol. XXII, pp. 593ff. The latter article, however, should be read with caution. The present writer has the best of reasons to believe that the marvelous correspondence in detail which exists between Bergson and the prince of pessimists is largely accidental. Bergson himself learned about it only after his own principles had been evolved into practically their mature shape.

[13] In the article in *The International Journal of Ethics* referred to.

either right or left, but simply "presses toward the mark." Of course there would be nothing in a mere illustration as such, but this one is so absolutely well chosen. This *is* the type of man—this steadfast man, this man who just is *not* sinuous and yielding and pliable and graceful and free, this straight-going individual who cannot do anything but go straight—*this* is the type whose proper destiny, according to the whole tenor of the essay, is to be laughed out of society; this is the man for whom society has no use. "Since when?" some may feel inclined to ask, not without a tinge of indignation. We confess that to us, hitherto, society has seemed to have considerable need for him; nay, to have had, perhaps, prodigiously little use for the other sort in comparison.

Moreover it is the discovery of precisely what this social theory neglects, namely the spirituality in spatiality itself, that enables the idealist to endorse the religious consciousness of God as eternal and perfect, without losing the other point, equally important, that the divine nature must also be movement, activity, freedom. To science the natural-spatial world is a completed order. If such order implies spirit, then, there must be a completed mind. As for the compatibility of such completeness with freedom, the very reasons which make Bergson to see real, active, free spiritual life except in a present which has the past in it, make it impossible for the idealist to see the perfection of such freedom except in a living present charged not only with the whole past, but with the whole future as well. The whole of reality must interpenetrate as Bergson makes the reality which has so far elapsed do. That interpenetration, with its inner activity, movement and freedom, makes up the content of what the religious consciousness has conceived as the perfect mind of God. Its inward intensity is God's perfect life, which is also ours so far as we are both good and great.

With the claim then, which is put forward by most of Bergson's following here and elsewhere that his philosophy is both true and "new," we cannot agree. So far as we have been able to examine it, it differs from other idealism in an essentially philosophical way only when it has something to say which is indefensible. Bergson has done important work in matters which in this paper we have had to pass over because they are extra-philosophical. He has done great work in psychology; and he has also done great work in the interpretation of the actual story of evolution, by bringing out new facts there which could easily be shown to be as compatible with the classical idealistic defense of spirit as with his own. That kind of work is the limit, it seems to us, of his service; except indeed it be a service to have presented a great deal of the substance of idealism from an angle so entirely fresh as almost to transport the reader into the idealistic center of vision, without his suspecting that he is there. We are not convinced that this is a small service. Nay, rightly understood, there is perhaps no greater.

J. W. Scott.

University of Glasgow.

THE PRESENT STATUS OF THE UNCONSCIOUS.

THE unconscious is a topic with which some writers have tried to coquet freely, which others have shunned scrupulously, and which still others have approached in a true scientific spirit in the endeavor to find out precisely what it is and how well it can explain the phenomena that usually come under its name. The motives that have led men to write about the unconscious have differed so widely that it is not surprising to find the works of some fantastical, and those of others useful for practical purposes only but devoid of scientific information. The interests of the former type have been purely metaphysical. Their object being to discover unity and continuity in the universe, they have postulated the unconscious as the absolute principle. The interests of the latter, who are chiefly physicians, have been entirely practical. Naturally they have considered and still continue to consider the subconscious from the functional point of view. What its real nature is and how it is related to consciousness as such, is a problem that does not fall within their sphere. The third group comprises the few psychologists who, motivated by a true scientific and progressive spirit, are seeking to discover the "what" and the "how" of subconscious activities. As a result of these diverse attitudes we find that one writer thought the unconscious a topic sufficiently great and all-embracing

UMI

to deserve three large volumes, while another laconically dismisses it in three monosyllabic words. But the fact that we still have the problem on our hands shows that neither the three volumes of von Hartmann nor the three words of Münsterberg[1] have either brought us any nearer to its solution or diminished in the least our unavoidable duty as scientific psychologists to search out the cause of unconscious activities and, if possible, to bridge the gap between the conscious and the unconscious.

As an example of the psychologist who avoids all discussion of the subconscious, I need only mention Titchener, who, after defining the subconscious as "an extension of the conscious beyond the limits of observation,"[2] goes on to say that it is always a matter of inference and therefore "it can not be a part of the subject matter of psychology." It is merely employed as an explanatory concept, he declares, but there are two reasons against its use in psychology: First, that the scientific psychologist, like the scientist in general, is not called upon to explain anything; and secondly, the introduction of this inferential concept may lead to danger inasmuch as it is "impossible to draw the line between legitimate and illegitimate inference."

As an example of the thinker who interprets all mental phenomena in terms of the subconscious, I may mention von Hartmann and Schopenhauer. The former endows the entire universe with an unconscious mind, declaring it to be the absolute principle which operates in all things organic and inorganic. But as James says, "his logic is so lax and his failure to consider the most obvious alternative so complete, that it would....be a waste of time to look at his arguments in detail." Nor are the views of Schopenhauer much more reasonable. According to him every sense organ unconsciously infers its impinging stim-

[1] *Psychotherapy*, p. 125.
[2] *A Beginner's Psychology*, p. 327.

ulus "as the only possible cause of some sensation which it unconsciously feels."[3]

But the theory of the subconscious dates back farther than von Hartmann and Schopenhauer. Although Weingärtner traces it to Plato and Plotinus, we may say that it received its first definite formulation at the hands of Leibniz in his conception of the *petites perceptions* which play the main role in psychic activity. These subliminal perceptions are individually too faint to arouse consciousness, according to Leibniz, but in their totality they come to a high degree of consciousness. To use his own words, "the belief that there are no other perceptions in the soul than those of which it is conscious, is a great source of error."

Kant's view of the subconscious is somewhat analogous to that of some modern authors, particularly Lipps. He declares: "To have sensations and not to be conscious of them is a contradiction, for how can we know that we have them, when we are not conscious of them? But we may infer we have had a sensation or a perception, although we were not immediately aware of it."[4] Such perceptions Kant calls "vague," and their field he declares is much broader than that of the clear and definite ones.

Turning to the English school, we find Sir William Hamilton asserting that although he does not wish to maintain that all consciousness is the product of unconscious perceptions, and that knowledge as such is the product of the unknown and the unknowable, still we must confess, he says, "that there are things which we neither know nor can know directly, but which manifest their existence indirectly through the medium of their effects."[5] Hence, since the mind in its behavior manifests processes of which it is unconscious, these processes must have come about through

[3] James, *Psychology*, I, p. 170.
[4] Soewenfeld, *Bewusstsein und psychisches Geschehen*, p. 2.
[5] Carpenter, *Principles of Physiology*, p. 518.

UMI

some modification of mind, and this may be called the unconscious. Or as Carpenter believes, Hamilton meant by this "unconscious cerebration."

Maudsley undertakes to interpret the greater part of conscious behavior in terms of unconscious psychophysical processes. He observes that almost from the moment of birth the sensorium receives multifarious impressions which it assimilates unconsciously, and makes use of them in a purely mechanical manner even in so-called intelligent activity. Even our general and abstract concepts are developed unconsciously; in short, "the process upon which our thinking depends," he says, "goes on of its own accord, without our awareness."

Carpenter devotes a whole chapter to unconscious cerebration and declares that since there is reason to believe that the greater part "of our intellectual activity"—whether it be reasoning or imagination—is essentially automatic, it is not unlikely that "the cerebrum may act upon impressions transmitted to it, and may elaborate intellectual results, such as might have been attained by the intentional direction of our minds to the subject, without any consciousness on our own part" (p. 515).

This view was subsequently taken up by Huxley and made to explain all intellectual activity. Noticing that epileptics can execute complex actions without having any memory of them upon recovery, and also that somnambulists can write letters and compose original verse while in their so-called sleeping state, he concluded that since "these cases are examples of purposive and intelligently controlled action taking place without consciousness, it would seem to follow that the mere mechanism of the nervous system" is all that is needed for the execution of such actions, independently of all consciousness and conscious guidance; and, therefore, we are compelled to assume that when similar actions are accompanied by con-

sciousness, the nervous mechanisms are the only essential conditions, and "consciousness is a superfluous accompaniment, so far as the causal sequence is concerned."[6]

It is obvious from the above that not only do earlier writers disagree on how the subconscious functions but they even differ with respect to its essential nature.

Turning to modern authors we find that among them, too, there are almost as many views as writers on the subject. On the one hand, men like Freud declare that subconscious phenomena are due to dissociated and suppressed ideas; that these unconscious ideas are active, though the individual may not be aware of them while going through the bodily actions of which those ideas are the prime causes.[7] On the other hand, Sidis retorts that the existence of unconscious ideas is inconceivable, for "ideas are essentially of a conscious nature";[8] hence their introduction into psychology is a self-contradictory concept. The subconscious, according to him, is rather "a diffused consciousness below the margin of personal consciousness."[9] Again, writers like Prince and James conceive of the subconscious as the outerlying fringe of consciousness, as dim consciousness, or better still, as the base of a cone, the apex of which is attentive consciousness. While Irving King refutes this view, declaring that consciousness either exists or does not exist, that "it may be more intense at one moment than at another.... But at any one moment it is.... a unitary existence without parts which might be thought of as clustering about a center with different degrees of intensity and adhesion."[10] Finally, while Sidis mocks unconscious cerebration, characterizing nerve currents, nerve-paths and

[6] McDougal, *Body and Mind*, pp. 109-110.

[7] Freud, "A Note on the Unconscious, "*Proc. Soc. Psy. Res.*, XXVI, 1912, p. 314.

[8] Sidis, "The Theory of the Unconscious," *Proc. Soc. Psy. Res.*, XXVI, 1912, p. 337.

[9] *Ibid.*, p. 319.

[10] "The Problem of the Subconscious," *Psychol. Rev.*, XIII, 1906, p. 43.

UMI

neurograms as "figments of imagination,"[11] Ribot maintains that the psychological aspects of the subconscious play but a secondary role, that they are a result, an effect of physiological or neural processes.[12]

No small part of the above controversy and disagreement is due to the fact that the term subconscious has been used in widely different senses. Prince gives no less than six different meanings in which the term has been employed.

1. The word subconscious has been employed to describe that portion of our field of consciousness which at any moment is outside the focus of attention. In this sense it is equivalent to James's fringe of consciousness.

2. The second meaning asserts that the subconscious is composed of ideas that are dissociated or split off from the personal consciousness, i. e., the focus of attention; that though the subject is unaware of their existence they are none the less active, and that "they form a consciousness coexisting with the primary consciousness, and thereby a doubling of conscious results."

3. According to the third meaning of the term, "subconscious states are conceived of as becoming synthesized among themselves, forming a larger self-conscious personality, to which the term self is given." These subconscious states are personified by the people who hold this view, and referred to as the "subconscious self" or "the hidden self."

4. The fourth view conceives the subconscious as "including all those past conscious states which are either forgotten and cannot be recalled, or which may be recalled as memories," their non-existence being due to the fact that they are crowded out of consciousness by the bulk of present experience.

[11] *Op. cit.*, p. 325.

[12] "A Symposium on the Subconscious," *Journ. Abnorm. Psychol.*, II, 1907, p. 37.

5. The fifth view, which is that of Frederick Myers, "declares that subconscious ideas, instead of being mental states dissociated from the main personality, are the main reservoir of consciousness, and the personal consciousness is a subordinate stream flowing out of this great storage." .....In short, we have within us a great tank of consciousness, but are aware of only a small portion of it.

6. The sixth and final view asserts that there are no psychical elements in subconscious phenomena at all, that automatic writing and speech, the solution of mathematical problems in sleep, and the carrying out of post-hypnotic suggestion "are the result of pure neural processes," unaccompanied by any mentation whatever.[13]

Before presenting the detailed arguments in support of the above theories, let us hastily review some of the more common subconscious phenomena in order that we may have freshly before our minds the facts which these theories endeavor to explain.

The main test that a by-gone experience was accompanied by consciousness is memory.[14] The ability to recall an experience without the artificial aid of suggestion or abstraction, shows that the individual was conscious of that experience at the time he underwent it. But memory is composed of three factors: registration, conservation and reproduction. Something must be impressed on the sensorium in order to be recalled, and it must also be conserved in some form. The question therefore arises: Does every impression, however faint it may be, stir up a pulse of consciousness which is immediately forgotten because of its brevity or faintness, or can reproducible impressions be made without the least awareness at the time being? And if so, how are they conserved? Daily observation and laboratory experiments demonstrate that perceptions of the

[13] "A Symposium on the Subconscious," *Jour. Abnorm. Psychol.*, p. 22.
[14] McDougal, *Op. cit.*, p. 109.

environment of which the individual did not have the least awareness, may be conserved. You may pass an acquaintance on the street without being aware of him at the time, but two or three minutes later it will suddenly dawn on you that you had seen your friend so and so. Again in hypnosis, by means of automatic writing or abstraction, people have been able to recall paragraphs in the newspapers read through casual glances without awareness thereof. Or the experiment may be put under controlled conditions, by having the subject take a brief survey of the room, and then while blindfolded dictate as detailed a description of it as he can. Thereafter if he is hypnotized and asked to describe the room once more, "it is often quite surprising," says Morton Prince, "to note with what detail the objects which almost entirely escaped conscious observation are subconsciously perceived and remembered."[15] Another method of proving the conservation of unconscious experiences is to have a person concentrate his attention by giving him something to read or an arithmetical problem to perform, and while he is so engaged to place cautiously and surreptitiously objects within his peripheral field of vision. After their removal he is asked to state in detail what he has seen. Invariably he is unable to mention any of these surreptitiously introduced objects. On being hypnotized, however, he mentions them with considerable accuracy and readiness.[16]

Automatic writing furnishes another group of facts which presuppose subconscious processes. If into the anesthetic hand of an hysterical person a pencil be put the hand will commence to write mechanically, and the subject will observe the movements of the hand as if that member belonged to some other person. Nor will the patient recognize the written ideas as his, but if he is hypnotized he

[15] Prince, *The Unconscious*, p. 53.
[16] *Ibid.*

will claim them immediately and explain what he meant by them. Sometimes the two hands of the same subject may be made to give written expression to two different kinds of mental content.

Perhaps the most interesting and common source of subconscious phenomena is somnambulism. People in this state have been known to perform the most delicate feats of physical skill, such as walking across roofs on narrow planks. Others have been known to perform events that in waking life require a great deal of intelligence,—such as writing letters or verse. Yet they have no memory for these events. The question arises: Are these highly complex mental activities performed mechanically, without any mentation, or are they consciously performed, but forgotten in waking life because dissociated from the personal consciousness?

Post-hypnotic suggestion is no less a mystery than complete change of personality. An individual is hypnotized and is told that at a fixed time after he awakens—be it several minutes, an hour or a day later—he is to do a certain deed. He is awakened and asked if he remembers anything that had been said to him during the hypnosis. He does not. He is permitted to depart and goes about his business in his customary manner. But precisely at the fixed time he will carry out the post-hypnotic suggestion, whether it be to ask for a pail of coal in a jewelry store or to purchase an overcoat in summer. When he is asked why he did this he can only reply that something within prompted him to it, that he felt it was a voluntary deed.

Whether such a case as that of the Rev. Ansel Bourne would fall into the group of epileptic phenomena or not matters little. In both instances we know that the subject will go through many complicated activities, denoting a high degree of consciousness or the presence of the customary kind of intelligence as judged by the adaptation of

the subject to his environment, yet in neither case does the normal personality have a memory for these experiences.

How are these phenomena to be explained in the light of modern psychology?

Two general theories are proposed: the psychical and the physiological. And it is to these two that the six foregoing views can be reduced after we eliminate Myers's metaphysical notion which conceives of the subconscious as the reservoir of all consciousness, and that other view which interprets the subconscious as the larger self-conscious personality.

Freud, Sidis and Janet may be taken as the chief exponents of the psychological theory of the subconscious, while Pierce, Jastrow and Ribot, not to mention a host of others, hold to the physiological view. The former trio in one form or another declare that the subconscious is dissociated consciousness, or awareness that is dissociated from the synthesizing personality, and that this awareness exists in consciousness in a latent form all the time. The latter maintain that not only is it unscientific to speak of latent ideas and latent feelings, but that there is no causal relation among psychic elements at all, that the explanation of unconscious phenomena must be sought in neural processes.

Let us examine their views individually.

Freud suggests that the term "conscious" should be applied to the perception which is present to our consciousness and of which we are aware,—while the latent perceptions should be denoted by the term "unconscious." "Hence an unconscious idea is one of which we are not aware, but the existence of which we are nevertheless ready to admit because of other proofs or signs."[17] This unconscious idea, though latent in the sense that it does not attain awareness, is by no means inactive while in the

[17] *Op. cit.*, p. 313ff.

mind. That unconscious ideas are active, undergoing combination and recombination among themselves, is demonstrated by the hysterical patient. "If she is executing the jerks and movements constituting her fit," says Freud, "she does not consciously represent to herself the intended actions, she may perceive those actions with the detached feelings of an onlooker. Nevertheless, analysis will show that she is acting her part in the characteristic reproduction of some incident in her life, the memory of which was unconsciously present during the attack."

Freud distinguishes two kinds of latent ideas: those which enter consciousness with no difficulty whatever, and those which do not penetrate into consciousness however strong they may be. The first type constitute the foreconscious, the second type the unconscious. "The term unconscious," he says, "now designates not only latent ideas in general, but especially ideas with a certain dynamic character, ideas keeping apart from consciousness, in spite of their intensity and activity." In explaining the phenomenon of double personality Freud would say that it is a shifting of consciousness, an oscillation between two different psychical complexes which become conscious and unconscious alternately.[18]

But the question still remains: Why does foreconscious activity pass into consciousness with no difficulty, while an unconscious activity is cut off from consciousness? (It is to be noticed here that he no longer speaks of foreconscious and unconscious ideas, but replaces the word idea by the term activity.) In answering this question he says that frequently when we try to represent an idea or a situation to ourselves we become aware of a distinct feeling of repulsion which must be overcome; and when we try to inject such an idea into a patient, we get the signs of what may be called his resistance to it. "So we learn that the un-

[18] *Ibid.*, p. 315.

conscious idea is excluded from consciousness by living forces, which oppose themselves to its reception; while they do not object to other ideas,—the foreconscious ones." At the present state of our knowledge, therefore, he suggests the following as the most probable theory that can be formulated: "The unconscious is a regular and inevitable phase in the processes constituting our psychical activity; every psychical act begins as an unconscious one, and it may either remain so, or go on developing into consciousness, according as it meets with resistance or not." Freud illustrates this view by referring to ordinary photography. The first stage of the photograph is the "negative"; every picture has to pass through the negative process; and those negatives which on examination prove to be satisfactory are admitted to the positive process, ending in the picture; those which do not are rejected. Such is the distinction between the foreconscious and unconscious ideas or activities. In reply to his critics that an unconscious idea is inconceivable, he declares that "the existence of an unconscious consciousness is still more objectionable."

Sidis gives three definitions of the subconscious which may be called the medico-popular, the metaphysical and the scientific, respectively. In one place he defines the subconscious "as mental processes of which the individual is not directly aware." In another place he refers to it "as a diffused consciousness below the margin of personal consciousness"; and on a third occasion he defines it "as consciousness below the threshold of attentive personal consciousness."[19]

The subconscious like the conscious may be, according to Sidis, of three types: desultory, synthetic, or recognitive.[20] Sidis would almost banish the term subconscious from literature, and what is commonly called subconscious he

[19] *The Theory of the Unconscious*, p. 319.
[20] *Psychology of Suggestion*, p. 201.

would call conscious, while that which is commonly known as the conscious he would call the self-conscious. The self-conscious is that form of mentation which is aware of itself; it is "the knowledge of consciousness within the same moment of consciousness." and in that sense it is identical with personality,[21] On the other hand, the secondary or subconscious self must not be regarded as an individual; "it is only a form of mental life"; it is a coordination of many series of moments-consciousness,"—i. e., pulses of consciousness. And it is these moments-consciousness that are at the heart of the subconscious. Therefore, subconscious experience is not *un*-conscious experience. The proof is this: Normal memory is a reproduction of conscious states. Now, when a subject is hypnotized he can be made to recall an experience which he does not remember in his waking state; and in this he displays memory like normal memory. Therefore, we have proof that his experience was accompanied by consciousness at the time it occurred. Or, to use Sidis's own words, "that in subconscious states there is really present a subconscious consciousness."[22]

It is to be noticed that this is not the same thing as saying that the ego or the personality was aware of that experience, but on the contrary, there was an awareness of which the attending self had no consciousness.

Having eliminated the subconscious from literature, therefore, there are only two forms of awareness to be considered, according to Sidis: consciousness as such and self-consciousness. The difference between these two states may be made clear in the words of Höffding. "Many feelings and impulses stir within us, without our clearly apprehending their nature and direction. A man who has this feeling does not know what is astir in him; perhaps others see it, or he himself gradually discovers it; but he

[21] *Ibid.*, p. 198.
[22] *The Theory of the Unconscious*, p. 331.

has the feeling that his conscious life is determined in a particular way."[23] What Höffding means is that there are "mental states of which we have consciousness, but which do not reach the personal consciousness." This is the distinction that Sidis makes between the subconscious and the self-conscious.

It naturally follows from the above, and there are many facts in support of the conclusion, that "the stream of subwaking consciousness is broader than that of the waking consciousness, so that the submerged subwaking self knows the life of the upper, primary self, but the latter does not know the former." He admits, however, that there are cases on record showing that the two streams may flow in separate channels; that the two selves may be ignorant of each other.[24]

On the basis of the foregoing view, the phenomenon of double personality is not difficult to explain, thinks the author. When a sufficient number of the submerged moments of consciousness have accumulated they tend to become synthesized, to group themselves in constellations and break forth into attentive consciousness, as do hallucinations, for example. In this manner the secondary consciousness attains self-consciousness, and appears as a new and independent personality. Now and then it "rises to the surface and assumes control over the current of life." This secondary self is aware of and passes judgment on the primary self, while the latter, when it returns, has not the least knowledge of the intruding ego.

It is apparent that the views of Freud and Sidis are essentially the same. The argument, therefore, that exists between these two writers is purely verbal and meaningless. There is no fundamental difference between an unconscious idea and an unconscious moment-consciousness, or even an

[23] Quoted by Sidis in *The Theory of the Unconscious*, p. 339.
[24] *Psychology of Suggestion*, p. 198.

unconscious consciousness. There may be a difference in quantity but not in quality. Yet we find Freud declaring that if philosophers find it difficult to accept the existence of unconscious ideas, the existence of unconscious consciousness is still more objectionable. To which Sidis retorts: "An idea is essentially of a conscious nature. To speak, therefore, of unconscious ideas is self-contradictory, —it is equivalent to the assumption of an unconscious consciousness."[25] I do not see why Sidis should find fault with this conclusion, since it is the very assumption with which he opens his own discussion on the theory of the subconscious. There he defines the subconscious as mental processes of which the individual is not aware. But what are mental processes if not ideas, images and perceptions? His definition, therefore, turns out to be precisely the same as Freud's.

Though the views of neither of these men lend themselves to acceptance in the light of the fundamental postulate of psychology,—namely, that every psychosis has its neurosis (but not the reverse), still Freud's doctrine of the subconscious is somewhat more palatable than that of Sidis. At least it is capable of interpretation in terms of our existing knowledge of neurology; it does not assume too much and does not pretend to offer a solution of all mental phenomena. The view of Sidis, on the other hand, is entirely out of harmony with the fundamental postulate of psychology, and it is so all-embracing and metaphysical in nature as almost to remind one of the teachings of von Hartmann.

This is demonstrated by the vigorous but wholly unjustifiable attack that Sidis launches against the theory of unconscious cerebration. This doctrine, it will be recalled, states that physiological processes may go on in the sensorium which enable the organism to adapt itself to its

[25] *The Theory of the Unconscious*, p. 337.

environment without any consciousness on its part. If this is so, says Sidis, then there is no reason why similar adaptations which are accompanied by consciousness should not also be purely mechanical and automatic. If the writing of letters during somnambulism is automatic, then the correspondence of waking life must be carried on in the same manner. But, he asks, "Can unconscious physiological processes write rational discourse? It is simply wonderful, incomprehensible." Assuming that every sense impression leaves behind it a trace, or a slight modification of nerve tissue, he says, still this does not explain why it is that a series of sensations, ideas, and images experienced at different times "should become combined, brought into a unity, felt.... like copies of one original experience."[26] Consequently the subconscious must be considered not as "an unconscious physiological automatism," but as "a secondary consciousness," as a secondary self.[27]

It is doubtful whether the theory of unconscious cerebration can account for the whole of unconscious phenomena, but there is no doubt that Sidis's notion does not account for even a fraction of it. The weakness of his logic is seen in such passages as the following: "Reactions to environment accompanied by intelligence in us are rightly judged to have the same accompaniment in others." From which, of course, he would have us draw the conclusion that since we guide our footsteps on the crowded street, or build a fire, with some degree of waking consciousness or intelligence, therefore the stroller who is absorbed in his newspaper or the somnambulist who builds a fire is also guided by awareness. This conclusion would be correct, provided the proposition on which it is based were not reversible. But it is reversible. It is precisely because we perform many so-called intelligent actions (as judged by

[26] *Psychology of Suggestion*, p. 125.
[27] *Ibid.*, p. 128.

their end product) without any consciousness in *our normal* life, that we rightly claim such actions to be devoid of intelligence or active consciousness in *other beings* when performed under the same conditions, or when those beings are abnormal. The above proposition, therefore, stands incomplete without its complement, which says with equal right: Reactions to environment not accompanied by intelligence and attentive consciousness in us are rightly judged to be devoid of these accompaniments in others, especially when those others can give no direct testimony as to the presence of consciousness.

Let us take an instance of so-called intelligent action which is accompanied without consciousness so far as memory can testify, and see whether it must be explained only on the basis of unconscious-consciousness, or whether a better explanation cannot be found. The case of the person who, though absorbed in his magazine, still picks his way through the crowded thoroughfare will do quite well. Now two wholly unrelated streams of thought cannot occupy the same mind at the same time. To be sure, we may dream and know that we are dreaming, or dream and experience a desire to wake up; or experience both the music and the color effect of an opera at the same time; but these are somewhat related mental complexes: at least they are logically related. We certainly can not solve mathematical problems and at the same time think of our social engagements. Suppose, then, we assume that our hypothetical person is strongly conscious of his reading material only, and is oblivious to the people on the sidewalk. How shall we explain his ability to pick his way through the crowd?

The process may be described thus: Two sorts of stimuli, diverse in nature, impinge on a single sensory organ, the eye. The one stimulus is the words on the printed page, which falls in the center of visual regard; the other stimulus is the people on the sidewalk, perceived in the periph-

UMI

ery of vision. Tracing these diverse impressions it seems reasonable to assume that the impression of the printed page is conducted to the occipital lobes, from there to the association centers, and from these the nerve energy is distributed to the other centers, including the motor center, so that when the individual reaches the bottom of the page he makes a conscious and coordinated movement with the hand to turn over a new page. The other vague impressions which fall on the periphery of vision are also conducted to the occipital lobes, but the path to the association centers is already blocked. Naturally the nerve energy seeks an outlet in some other direction. Now in the course of the individual's life, strong association bonds had been formed between visual perceptions of the kind that now impinge on the periphery of his vision and specific organic reactions, i. e., seeing a body coming toward him and moving out of its way. Psychophysically speaking, these strong association bonds are smoothly working conduction-paths between the visual and motor centers. Consequently when now a visual impression of the same kind reaches the visual center, it immediately discharges itself through the path of least resistance, and upon reaching the motor center releases the customary response which, of course, is an adaptation to the external situation. Since all this takes place without reaching the association centers, we have unconscious "intelligent" action.

But it will be asked: How does this view account for the fact that if the individual is hypnotized he can be made to give an account of persons and places he had passed though wholly oblivious of them at the time? The answer to this question involves the physiological theory of the unconscious, and it is to this that we turn next.

Generally stated, this theory means that the subconscious is not psychical at all, but purely physiological; that the presence of awareness cannot be measured by adaptive-

ness of action, for there are many glands and thousands of cells in the human body performing very complex adaptive acts, or acts designed for the preservation of the organism; yet we do not say that these are mental. Why should we expect less from the tissue of the central nervous system than we do of all other tissue? Or in the words of Münsterberg, "Why cannot they, too, produce physiological processes that yield to well-adjusted results?," i. e., to purposive sensorial excitements and motor impulses.[28]

The same view is advanced by Ribot, who declares that the psychological solution of the unconscious rests on the assumption that consciousness is a quantity which may decrease indefinitely without ever reaching zero. But there is no justification for this postulate; he says: "The results of psychophysists with regard to the threshold of consciousness seem to justify the opposite view, namely, the perceptible minimum appears and disappears instantaneously, and this fact is unfavorable to the hypothesis of an increasing and decreasing continuity of consciousness." The physiologic solution, moreover, is simple, inasmuch as it maintains that subconscious activity is purely cerebral.[29]

The same theory is shared by Jastrow. He deems it a fundamental requisite of any adequate conception of the subconscious that it make a vital connection with normal mental activity; it must find a natural place in an evolutionary interpretation of psychic functions, and like normal activity it must be interpreted in terms of neural dispositions. He proposes a criterion, therefore, for the measure of awareness. "The measure of awareness that shall accrue to any given nervous structure to an environmental situation, in order to render the response advantageous... will be determined by the status of the need thus satisfied in the organic life of the individual. The simplest, recur-

[28] *Journ. Abnorm. Psychol.*, II, 1907, p. 30.
[29] *Op. cit.*, p. 35.

rent and constant needs will be sufficiently met by neural dispositions without conscious states, or with the lowest type thereof."[30]

Irving King advances the same view and almost in the same words. "Neural processes," he says, "are accompanied by psychical processes only when there is some need for them."[31] According to him, consciousness is definitely related to the facilitation of reactions and adjustments required by the life process, but which the automatic arrangements of the organism cannot meet. Consciousness either is or is not. It may be more intense at one moment than at another, but it does not consist of different degrees of intensity, as James's theory of the "fringe" would imply. On the neural side, however, we do have a system which may be spatially represented. In terms of this system consciousness is not "the sum of the organization of psychic elements, but rather the unique and single accompaniment of a peculiar organization of neural processes." From this definition it follows that each neural element will determine the complexion of consciousness. If it is in the center of the system, it has dynamic conscious value; if outside of that system, it has potential value only. The subconscious, therefore, is not to be conceived as dim consciousness, but rather as a "physical mass of neural dispositions, tensions and actual processes which are in some degree, perhaps organized, the remnants of habits and experiences, both those which have lapsed from consciousness and those which have never penetrated the central plexus."

On the basis of these definitions it becomes fairly easy to understand most of the phenomena that come under the heads of the conscious and the unconscious. "When consciousness is present," say King, "the neural processes involved are much more intense than otherwise." The dream consciousness is a condition in which the central activity

[30] *The Unconscious*, p. 411. [31] *Op. cit.*, p. 42.

is so subdued that more or less fragmentary neural dispositions are aroused. In hypnosis, again, the center of activity is shifted in more or less degree, resulting in the temporary lapse from consciousness of some processes and the incorporation of others which were previously mere neural dispositions. While in multiple personality there are one or more strongly organized potential systems of neural elements which, under appropriate conditions, can separately become sufficiently active to be conscious.[32]

It is to be noted that the chief characteristic of the exponents of the physiologic theory is that they do not endow the subconscious with any mysterious powers, they do not regard it as the reservoir of consciousness, but on the contrary, they consider subconscious events as very much like the ordinary facts of waking consciousness; and their method of explanation is to proceed in a true scientific manner from the known to the unknown, from the facts of the conscious to those of the unconscious. And although Morton Prince does not hold this view in its entirety, it is nevertheless in this fashion that he commences the presentation of what is without doubt the most able and most cogent theory of the unconscious that has appeared in recent years.

The problem of the subconscious, according to him, is the problem of memory. Whoever solves the latter will also have solved the former. Memory should be considered from two points of view: as a process and as an end result. As a process it is composed of three factors,—registration, conservation and reproduction. The last is the end result, but to understand this we must know something of, or at least have a plausible theory concerning, impression and conservation.

Instances of the conservation of forgotten experiences abound both in normal and pathological life. They are

[32] *Ibid.*, pp. 45ff.

such as lapses of memory, forgotten acts, failure to recognize, or in abnormal cases they become manifest in automatic writing and speech, in post-hypnotic suggestions, and so forth. After examining the facts in great detail, Prince comes to the conclusion that it does not matter at what period of life or in what state experiences have occurred, "or how long a time has intervened since their occurrence, they may still be conserved. They become dormant, but under favorable conditions, they may be awakened and may enter conscious life."[33] Naturally these experiences must be conserved in some form; and whatever the nature of this form may be it is obvious that the experiences themselves must have "a very specific and independent existence, somewhere and somehow, outside of the awareness of consciousness."

Now in order to account for normal memory we must posit that ideas which have passed through the mind have been conserved through some residuum left by the original experience. This residuum must be either psychological or physiological. Suppose we consider the former alternative first. We shall have to assume that sensations, perceptions, emotions and even complex systems of ideas are capable of pursuing "autonomous and contemporaneous activity outside of the various systems of ideas that make up the personal consciousness." This is an untenable view, for it would necessitate the storing up of millions of ideas and infinite forms of associations. Let us, therefore, consider the other alternative, namely, conservation as physical residua. This view is based on the assumption that whenever we have a mental experience of any sort some change or trace is left in the neurones of the brain. This does not necessarily mean that the neural modification is the cause of the conscious process. On the contrary, it assumes the postulate of psychophysical parallelism and

[33] *The Unconscious*, pp. 82ff.

declares that with every passing state of conscious experience, with every idea, emotion and perception, the brain process that is functioning leaves some trace, some residua of itself within the neurones and in the functional arrangements among them. This physiological conception is at the basis of the association theory, wherein it is assumed "that whenever a number of neurones involved in a coordinated sensory-motor act are stimulated into functional activity, they become so associated and the paths between them become so opened or sensitized, that a disposition becomes established for the whole group to function together and to reproduce the original reaction when either one or the other is afterward stimulated into activity. This 'disposition' is spoken of in physiological language as a lowering of the threshold of excitability. This change we may speak of as a residuum,"[34] says Prince.

We are now in a position to answer the question raised a while ago concerning the ability of a hypnotized person to recall a forgotten experience or one that he was not aware of at the time of its occurrence.

The neurones in retaining the residua of the original process have become organized into a functioning system corresponding to the system of mental states—whether ideas, perceptions or emotions—which accompanied that original experience and are now capable of reproducing it. Hence when we reproduce the original ideas in the form of memory it is because there is a refunctioning of the physiological neural process. On hypnotizing a person, therefore, and asking him to recall a forgotten event, we simply start that process by introducing what may be called a catalytic agent, i. e., we stir one neurone or one brain cell, or one part of the system, and that sets the entire system working precisely as it did on the original occasion. This physiological functioning now reaches consciousness or motor expression,

[34] *Ibid.*, pp. 119-120.

UMI

because all other mental processes are arrested for the time being, thus facilitating a greater discharge of nerve energy in this one direction.

The same is true of crystal gazing and automatic writing. In the former occurrence there is an intense concentration of primary attention. That is, the subject does not attend to any idea or to a situation from which he tries to derive meaning, but merely to a visual stimulus. In this manner all distracting influences and mental processes which do not harmonize with the original experience, of which it must be said the individual has some intimation to begin with, are arrested. Thus the resumption of the original neural process is facilitated and with it, of course, the psychical accompaniment. Anything that will hold the attention will do as well as a crystal. A soft light will work just as well.

Equally well can automatic writing be explained on the basis of this theory. The writing habit is very highly and delicately "developed in us writing mortals," to use a phrase of Pierce, and it is no wonder that it may operate mechanically, when for some reason its neural system has become detached from that other system which constitutes self-consciousness. Nor do the specimens of automatic writing show this phenomenon to be essentially different from the uncontrolled movements of the hands and bodily twitching that most of us have at times; and by no means is it different from such nervous troubles as chorea and locomotor ataxia. The hand has been observed to write backwards, to write mirror script, to follow indefinitely a direction given to it by the experimenter such as moving in a circle, it misplaces and omits letters. "Surely," says Pierce, "such occurrences point clearly to a disordered neural mechanism, rather than to a perverse or humorously inclined secondary consciousness."[35]

[35] Carman, *Studies in Phil. and Psychol.*, 1906, pp. 327-328.

We see, then, that most if not all subconscious phenomena can best be explained in terms of cerebration. Now it is necessary to have some term to designate the separate neurological modifications, and Prince calls these "neurograms." A neurogram, therefore, is a brain record; and, just as a phonogram characterizes the form in which the physical aspect of spoken thought is recorded, so a neurogram characterizes the form in which thoughts and other mental experiences are recorded in the brain tissue. Of course this is merely a theoretical concept, like atoms and moments of force.

Though memory is regarded in psychology as a conscious process, it is evident that on the basis of the foregoing view any process that consists of the three factors, registration, conservation and reproduction of experiences, must be considered as memory, "whether the final result be the production of a conscious experience or of one to which no consciousness was ever attached."[36]

That memory is ultimately a physiological phenomenon was demonstrated by the experiments of Rothmann who showed that decorticated animals can be educated, i. e., new dispositions and new associations may be established in the lower centers "without the intervention of the integrating influence of the cortex or conscious intelligence."[37] The bearing of this fact is that unconscious processes are capable of being conserved in the form of physiological memory.

If we accept the psychophysiological theory of memory, then, we may define the unconscious as the brain residua, the physiological dispositions or neurograms in which the experiences of life are conserved. The co-conscious, on the other hand, means "a coexisting consciousness of which the personal consciousness is not aware. And since these

[36] Prince, *The Unconscious*, p. 135.
[37] *The Unconscious*, p. 238.

two function together we need an inclusive term, one that will embrace them both, and that is the subconscious."[38]

Here the truly scientific discussion ends. The rest that Prince has to say about the subconscious is metaphysical, and not unlike the views of Sidis, von Hartmann and Myers. He declares, for instance, that the subconscious, rather than the conscious, is the important factor in personality and intelligence; that the subconscious furnishes the material out of which our judgments and beliefs, our ideals and characters, are shaped. Yet I can hardly see how he squares this statement with the next in which he says that the unconscious complexes are kept in check by the normal inhibitions and the counterbalancing influences of the normal mental mechanism.[39] Evidently, then, it is the normal mental mechanism, by which I suppose he means attentive consciousness or intelligence, which exercises a determining control over the unconscious complexes. Hence it is the conscious and not the unconscious which is at the basis of our beliefs, our ideals and character.

Be this as it may, Prince's metaphysical interpretation does not change the facts nor the value of his scientific concepts which so excellently explain those facts. For by resolving the subconscious into unconscious physiological dispositions on the one hand, and coactive conscious states on the other, we are able to understand more clearly the nature of lapsed memory, absent-mindedness, post-hypnotic suggestion, artificial hallucinations, hysteria, psychoneurosis and multiple personality.

With respect to bridging the gap between the conscious and the subconscious, Prince declares that no gap exists. What belongs to one at times passes into the other, and *vice versa*. Consciousness may be conceived of as a round disk with attention or the focus of awareness at the center.

[38] *Ibid.*, p. 253.
[39] *Ibid.*, p. 262.

Surrounding this is a zone which constitutes the fringe of awareness. Embracing that is the co-conscious, i. e., unconscious mentation, while the outermost zone comprises the unconscious processes. There is a gradual shading from the center to the edge of this figurative disk or sphere of consciousness. But here again Prince treads on the metaphysical, and we have not the time to follow him.

The space at our disposal only permits us to suggest that more original work ought to be done in this field. Too many writers weave their theories around the same cases of somnambulism and double personality. The cases examined by Morton Prince, Janet and Bernheim constitute a sort of stock-in-trade making their rounds in the literature on the subconscious. But a theory does not gain credence by hopping about on the same crutches; it must gather new facts if it would increase in strength. In this respect Professor Lillien Martin has shown a good way in her experimental investigation of the subconscious. She does not add anything new, but her method of investigation which consisted in having normal subjects permit images to arise of themselves and then introspect on them, is more reliable than the questionnaire method used by some authors, or the observation of pathological cases upon which still others have built their concepts. Experimental research under strictly controlled conditions, should be the slogan of psychologists in the field of the subconscious as it is in that of the conscious.

Thus we stand at the present moment with three theories of the unconscious before us. The psychometaphysical, the psychophysiological with metaphysical leanings, and the psychoneurological with scientific leanings. The first declares that the whole universe is permeated with consciousness, that there is intelligence in all animals, in plants and even in inorganic matter. This notion is held by writers like von Hartmann, Myers, Delboeuf and persons in-

UMI

terested in psychical research. It is obvious, however, that this view will not bring us anywhere.

The psychophysiological theory with metaphysical leanings also maintains that there is consciousness in all organisms, only it is not conscious of itself. That in living organisms this consciousness is accompanied by physiological changes, but these changes are not necessarily the cause of conscious phenomena. Neurological modifications are only conceptions assumed for the purpose of explaining unconscious activity. But the psyche is the fundamental principle. The unconscious is the source of all intelligence. This view is held explicitly or implicitly by Freud, Sidis, Prince, Lloyd Morgan and Janet.

Finally, the psychophysiological theory with scientific leanings asserts that neurological modifications are the essential factors of conscious and unconscious phenomena. That consciousness appears only when the neurones attain a certain tension, or function in a certain relation; that consciousness may or may not accompany so-called intelligent actions performed under pathological conditions; that it is certainly not present in instinctive functioning which characterizes the life of lower animals; that the unconscious is not the storehouse of the conscious, that there is nothing mysterious or wonderful about it, and that with further investigation its precise nature and place in the scale of psychogenesis will be at the command of psychologists. This view is held by writers like Ribot, Pierce, King and Jastrow.

Such are the three views that present themselves for our consideration. There is no doubt about the one that scientists will adopt as leading to a greater extension of human knowledge.

GUSTAVE A. FEINGOLD.

HARTFORD, CONN.

NIRVANA

THE BUDDHIST'S FINAL GOAL.

NIRVANA, state of rest unbroken, where
Benign extinction of all passion rules—
A rest so deep that in eternity
It shall not be disturbed—I long for thee!
After life's storm and stress thou grantest peace.
Weary of this world's wild anxieties,
Its pains and empty pleasures, I will seek
The everlasting in blank vacancy,
Thus to attain the boon of dreamless sleep
From which nor rancor of a villainous
Intrigue, planned by malevolence or hate,
Nor the misfortune of a sorry slip
Of my misguided feet, will waken me,
But unconcerned and calm I shall remain
In perfect quietude. For I'll be safe
From all the worry and from all the trouble
That restlessly stirs life and keeps it moving.
The bustle of the world, its vulgar noise
With its deplorable afflictions, trials
And eke malicious slander, will be hushed.

There is a refuge, vainly sought for here,
And in its sanctuary I'll find shelter
From life's great woes and small annoyances.
There paltry problems will no longer vex
Nor will demand immediate solution.

UMI

I shall no longer be disquieted
By urgent needs to be responded to
In energetic action. E'en my ego
With its ambitions, wants and vanities;
Its recollection of the past with all
Its sweet and bitter memories—all that,
My very consciousness, will be extinct.
I shall be left in tranquil emptiness
And in a soothing void of non-existence,
A clean, pure state of rest most absolute,
Without the slightest ripple of disturbance,
A panacea for all earthly ills,
An anodyne for any pang or pain.

In former ages mankind felt assured
Of a survival of the soul. The savage
Met his dead parents and his friends in dream.
He saw them, he conversed with them, and dreams
Were real to him just as actual life.
When man grew wiser, he began to doubt
And he grew anxious for convincing proof.
Though proofs were negative, yet he still clung
To hope expressing his desire to live
And to prolong his life beyond the grave.

Oh foolish man, why dost thou shrink from death
And yearnest greedily for prolongation
Of thy ill-favored self? Thy selfishness
Thou wishest to preserve, thy abject foibles,
Instead of gladly hiding them into
The darkness of a taciturn forgetting,
As in wise justice Nature has intended,
Thou wouldst perpetuate with petulance
And peevish childishness that of thyself,
Exactly that, whose riddance should be welcomed

As a great boon, a seemly liberation
From slavery, its drudgery and curse.
Why should we cling to chains that burden us
When we might cast them off and free ourselves?
Why should we serve new terms as sentenced convicts
When duly our acquittal is pronounced
And a reprieve has graciously been granted?

Mara, the Evil One will envy me
In my benign repose; he will continue
His vicious persecution. Shall I help him
And do the wrong myself unto myself
By pressing from a place of safety into
My prison with its ugly bars? Oh no!
No, I shall not! For I prefer my freedom!
There I shall be where most malignant foes
Shall not be able to do any harm.
And if they should go on abusing me
I shall no longer heed their defamation;
I'll leave them to their fate which in full justice
Will come to them without my interference.
Their lies no longer touch me. In Nirvana
I shall be free; the vicious will remain
In a gehenna builded by themselves.

The wild desires of my hot pulsing heart
Will then be calmed, all hunger will be stilled,
All thirst be quenched in deepest satisfaction.
And mine shall be the glory of Nirvana;
Having achieved the conquest of all pain,
Having attained final emancipation,
It will be mine, Nirvana will be mine.
I shall be free when I have closed mine eyes,
To enter death, life's solemn grand finale,
Its fruitage, benison and consummation.
Nirvana's holy peace shall then be mine.

UMI

Indeed it is mine here; I live it now
If I but understand the art of living
The truth: It is and will be mine, when I
Surrender transient things to transiency
And live in that alone which will endure.

Oh, the inanities of self! how puny,
How paltry are they; and how kind is death
To brush them off with gently sweeping stroke
Like spider webs out of a gloomy corner,
Together with the spider who has built them.

O let them go without regret and sorrow,
The ego with its portion is not worthy
Of preservation. It is but the burden
Of our existence, the receptacle
In which the weaknesses and faults of life
Are bred, in which its plagues are caught and stored.
So let them go and bless their disappearance.
They are like painful sores that should be healed,
And when our ego passes they are cured.
The right ideas only which we've thought,
The good deeds too which we have done and things
Of beauty we have shaped, they shall survive.
They are our better selves; they will be helpful,
Helpful to others, to the generations
That are to come, helpful like gifts of God,
Like rain or sunbeams, showered down on earth,
Profuse, unstinted, and with utter lack
Of egotism. But do not cling to self,
Nor yearn for any undue preservation
Of personality. Our ego's life
And all that's of an accidental nature
Be handed over to its destination
Which is a dissolution into naught.

Our conscious ego has originated,
It has been growing, and 'twill pass away.
Such is its destiny and so 'tis best.
But I will glory in my future lot—
Nirvana's boon, the state of perfect peace.

Yea, I can enter even now into
Nirvana's hallowed temple where my soul
Is liberated from all transiency
And will be ready for a final exit
Out of existence with its narrowness
Into the better and superior realm,
The realm of bliss, Nirvana's noble bliss.

Praised be Nirvana, glorious radiant state
Of biding peace, hope of all living creatures
And comfort of the dead. Holy asylum
Which grander is than highest joy in heaven
And more divine than the divinity
Of Brahma and his gods in all their splendor.
Praised be Nirvana, goal of all the Buddhas!
And blest is he who enters there, who lives
There in Nirvana; lives there in the truth
Which therein is revealed; he who is free
From vain attachment, who's above temptation.
'Tis he in whom all passion is extinct;
Who has attained life's final aim Nirvana,
Goal of the wise, and of the blessèd Buddhas.
He who has reached it is the Conqueror,
The conqueror of Evil, the great Jina,
He's the Enlightened One: he is the Buddha!
And he is blest; the Buddha, yea! is blest.
Pathfinder to Nirvana! Praised be he!
Namo tassa Bhagavato Buddhassa.

PAUL CARUS.

THE MANUSCRIPTS OF LEIBNIZ ON HIS DISCOVERY OF THE DIFFERENTIAL CALCULUS.*

PART II.

§ III.

The following notes, on certain MSS. which Gerhardt does not give in full, are taken from G. 1848, p. 20 et seq. (see also G. 1855, p. 55 et seq.)

In a manuscript of August, 1673, bearing the title *Methodus nova investigandi Tangentes linearum curvarum ex datis applicatis, vel contra Applicatis ex datis productis, reductis, tangentibus, perpendicularibus, secantibus,* Leibniz begins at once with an attempt to find a method that is applicable to any curve for the determination of its tangent. "But if," says Leibniz with regard to the classification of curves which Descartes laid down as fundamental for his method of tangents, "the figure is not geometrical — such as the cycloid—it does not matter; for it will be treated as an example of a geometrical curve, by supposing that there is a relation between the straight lines and curves by which they are made known to us; in this way, tangents can be drawn just as well to either geometrical or ageometrical curves, as far as the nature of the figure allows." He considers the curve as a polygon with an infinite number of sides, and here already he constructs what he calls the "Characteristic Triangle," whose sides are an infinitely small arc of the curve, and the differences between the ordinates and between the abscissae; this is similar to the triangle whose sides are the tangent, the subtangent and the ordinate for the point of contact. In just the same manner as used by Descartes, Leibniz seeks the tangent by means

* Part I appeared in *The Monist* of October, 1916.

of the subtangent; he denotes the infinitely small differences of the abscissae by *b*, and verifies for the parabola, that his method works out correctly, when the terms of the equation that contain the infinitely small quantities are neglected. The omission of these terms, however, does not appear to Leibniz to be a method to be relied upon. In fact, he says: "It is not safe to reject multiples of the infinitely small part *b*, and other things; for it may happen that through the compensation of these with others,[1] the equation may come to a totally different condition." So he seeks to obtain the determination of the subtangent in some other way. "The whole question is, how the applied lines can be found from the differences of two applied lines," are his own words. He then finds that the solution of this problem reduces to the summation of a series, of which the terms are the differences of consecutive abscissae.

At the end of the manuscript Leibniz proceeds to speak of the inverse problem: "It is an important subject for investigation, whether it is possible, by retracing our steps, to proceed from tangents and other functions to ordinates. The matter will be most accurately investigated by tables[2] of equations; in this way we may find out in how many ways some one equation may be produced from others, and from that which of them should be chosen in any case. This is, as it were, an analysis of the analysis itself, but if that is done it forms the fundamental of human science, as far as this kind of things is concerned." Ultimately Leibniz obtains the following result: "The two questions, the first that of finding the description of the curve from its elements, the second that of finding the figure from the given differences, both reduce to the same thing. From this fact it can be taken that almost the whole of the theory of the inverse method of tangents is reducible to quadratures."

According to this, Leibniz has in the middle of the year 1673 already attained to the knowledge that the direct and the so-called inverse tangent-problem have an undoubted connection with one another; he has an idea that the latter may be capable of reduction to a quadrature (i. e., to a summation).

Again, in a manuscript dated October 1674, i. e., fourteen months later, which bears the title *Schediasma de Methodo Tan-*

[1] It is impossible to see, without a fuller knowledge of the context, whether this refers to "compensation of errors," or whether Leibniz is alluding to the possibility of all the finite terms cancelling one another.

[2] Leibniz comes back to this point later; see § IV.

gentium inversa ad circulum applicata, he is able to say for certain that "the quadratures of all figures follow from the inverse method of tangents, and thus the whole science of sums and quadratures can be reduced to analysis, a thing that nobody even had any hopes of before."

After Leibniz thus recognized the identity between the inverse tangent-problem, of which the general solution had not been found by Descartes, and the quadrature of curves, he applied himself to the investigation of series by the summation of which quadratures were then obtained. In a very extensive discussion, bearing the date of October, 1674, and the title *Schediasma de serierum summis, et seriebus quadraticibus,* Leibniz starts from the series

$$\frac{b}{1} - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \frac{b^5}{5} - \cdots\cdots,$$

and obtains the following general rule: "By calling the variable ordinates x, and the variable abscissae y, and b the abscissa of the greatest ordinate e, and d the abscissa of the least ordinate h," are Leibniz's own words, "we have the following rules:

$$\frac{x^2}{2} = ywx - \frac{yw^2}{2} + \frac{d^2h}{2},$$

$$\frac{h^2w}{2} + \frac{d^2h}{2} = xy - \frac{x}{2},\ e-h=w,$$

$$xw = \frac{e^2}{2} - \frac{w^2}{2},$$

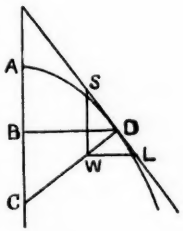
$yw = x$ in decreasing values, for in ascending or increasing values $yw = eb - x$."[3]

Leibniz then goes on to remark: "These rules are to be altered slightly according as the series increase or decrease; also mention of the least ordinate may be omitted, if it is always understood to be the last ordinate; on the other hand, w can always be inserted wherever mention is made of w. All series hitherto found are contained in the one by means of these rules, except the series of powers, which is to be obtained by taking differences."

[3] This, without either proof or figure, is a hopeless muddle; and yet it is repeated word for word, without any addition or remark, in Gerhardt's 1855 publication. Goodness knows what the use of it was supposed to be in this form! Unless Leibniz has omitted some length, which he has supposed to be unity, the dimensions are all wrong.

In the same essay, Leibniz makes use of a theorem, which he has probably found to be general at an earlier date, namely:

"Since BC is to BD as WL to SW, therefore BC⌒SW,[4] that is, the sum of every BC [applied to AC], is equal to BD⌒WL, that is, the sum of every BD applied to the base; moreover, the sum of every BD applied to the base is equal to half the square on the greatest BD. Further, it is evident that the sum of every WL is equal to the greatest BD."



Accordingly, Leibniz comes to the further conclusion that the method of Descartes, which uses a subsidiary equation with two equal roots, to solve the general inverse-tangent problem, is unsatisfactory. In a manuscript of January, 1675, Leibniz says: "Thus at last I am free from the unprofitable hope of finding sums of series and quadratures of figures by means of a pair of equal roots, and I have discovered the reason why this argument cannot be used; this has worried me for quite long enough."[5]

§ IV.

The manuscript that comes next in date is one that is given in G. 1855. It really consists of three short notes, (1) a theorem on moments, (2) a continuation of the idea started at the end of the manuscript of August, 1673 (§ III), namely the formation of tables of equations that are derivable from certain standard equations, with the appropriate substitutions for each case, (3) a return to the consideration of moments.

This is the first appearance of the word "moment," but from the context it is evident that Leibniz has done some considerable amount of work upon the idea before. If the theorem that is first given is written in modern notation,

[4] The sign ⌒ signifies multiplication.

[5] Observe that as yet nothing has been said about the area of surfaces of revolution or moments about the axis, although we should expect them to be mentioned in connection with the figure that is given; for the next manuscript shows that in October 1675, Leibniz has already done a considerable amount of work on moments.

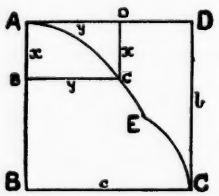
it takes the form of an "integration by parts" and serves to change the independent variable. Thus we have

$$\int xy\,dx = \left[\frac{x^2y}{2}\right] - \int \frac{x^2}{2}\,dy;$$

and it is readily seen that if x can be expressed as a square root of a simple function of y, as for the circle and the conic sections, then the integral on the right-hand side has no irrationality. This, I take it, is the connection between this theorem and those which follow.

The proof is not so clear as it might be on account of two errors, both I think errors of transcription or misprints. The first *a* should be an *x*, and the second *a* should be the preposition *a* (= from); also, for modern readers the figure might be improved by showing the *variable* lines AB (= x), BC (= y) as in the accompanying diagram. The argument then is as follows:

Moment of BC(=y) about AD is xy, when it is applied to AB for the summation; for this brings in the infinitesimal breadth of the line.



Moment of DE (= x) about AD is $x^2/2$, when applied to AD, so as to include the infinitesimal breadth of the line, and assuming that the line may be considered to be condensed at its center of gravity. The theorem follows at once.

Note the use of the sign ⊓ as a symbol of equality, which I have allowed to stand in the opening paragraph. Leibniz adopts the ordinary sign two months later, or Ger-

hardt makes the change,[6] so I have not thought it necessary to adhere to it, but only to show it in the opening paragraph.

The only remark that seems to be necessary with regard to the second part of this manuscript is that Weissenborn[7] argues from the continued allusion by Leibniz to the desirability of forming tables of curves whose quadratures may be derived from those of others, especially the conic sections, (starting with the manuscript of November, 1675, where Weissenborn states that it is first hinted), that Leibniz had probably either seen or heard of the *Catalogus curvarum ad conicas sectiones relatarum* of Newton. The point is that Weissenborn seems to have missed the clear reference to the reduction of curves to those of the second degree, in this manuscript of October, 1675. It may of course be just possible that G. 1855, in which this MS. appears, was not at Weissenborn's hand at the time that he wrote, for Weissenborn's book was published in 1856.

With regard to the third part, it will be found in the original Latin that Leibniz, after apparently starting with perfect clearness, gets rather into a muddle toward the end. This is however only apparent, being partly due to an inaccurate figure, and partly to what I am convinced is an error of transcription. This incorrect sentence makes Leibniz write apparently absolute nonsense; but if a correction is made according to the suggestion in the footnote, and reference is made to the corrected diagram that I have added on the right of the figure of Leibniz, as given by Gerhardt, then the proof given by Leibniz reads perfectly smoothly and sensibly.

[6] Gerhardt has a footnote to the effect that, as nearly as possible he has retained the exact form of this and the manuscripts that immediately follow; except in the matter of this one sign I have adhered to the form given by Leibniz.

[7] Weissenborn, *Principien der höheren Analysis*, Halle, 1856.

UMI

25 October, 1675.

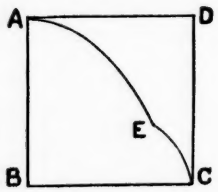
Analysis Tetragonistica Ex Centrobarycis.

Analytical quadrature by means of centers of gravity.

Let any curve AEC be referred to a right angle BAD; let AB⊓DC⊓a,[8] and let the last $x \sqcap b$; also let BC⊓AD⊓y, and the last $y \sqcap c$. Then it is plain that

$$\text{omn. } \overline{yx \text{ to } x} = \frac{b^2c}{2} - \text{omn. } \overline{\frac{x}{2} \text{ to } y}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

For, the moment of the space ABCEA about AD is made up of rectangles contained by BC ($= y$) and AB ($= x$); also the moment



about AD of the space ADCEA, the complement of the former is made up of the sum of the squares on DC halved $\left(= \frac{x^2}{2}\right)$; and if this moment is taken away from the whole moment of the rectangle ABCD about AD, i. e., from c into omn. x,[9] or from $\frac{b^2c}{2}$, there will remain the moment of the space ABCEA. Hence the equation that I gave is obtained; and, by rearranging it, it follows that

$$\text{omn. } yx \text{ to } x + \text{omn. } \frac{x^2}{2} \text{ to } y = \frac{b^2c}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In this way we obtain the quadrature of the two joined in one in every case; and this is the fundamental theorem in the center of gravity method.

Let the equation expressing the nature of the curve be

$$ay^2 + bx^2 + cxy + dx + ey + f = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and suppose that $xy = z$, $\ldots\ldots (4)$, then $y = \frac{z}{x}$. $\ldots\ldots\ldots\ldots (5)$

Substituting this value in equation (3), we have

[8] This a should be x.

[9] Here, in the Latin, "ac in omn.x" should be "a c in omn.x."

$$\frac{az^2}{x^2}+bx^2+cz+dx+\frac{ez}{x}+f=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and, on removing the fractions,

$$az^2+bx^4+cx^2z+dx^3+exz+fx^2=0. \quad \ldots\ldots\ldots\ldots (7)$$

Again, let $x^2=2w$ ....(8); then, substituting this value in equation (3), we have

$$ay^2+2bw+cxy+dx+ey+f=0, \quad \ldots\ldots\ldots\ldots\ldots (9)$$

and therefore

$$x=\frac{-ay^2-2bw-ey-f}{cy+d}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$=\sqrt{2w}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

and, squaring each side, we have[10]

$$a^2y^2+4aby^2w+2aey^3+2afy^2+4b^2w^2+4bewy+4bfw$$
$$+e^2y^2+2fey+f^2-2c^2y^2w-4cdyw-2d^2w=0. \quad \ldots (12)$$

Now, if a curve is described according to equation (7), and also another according to equation (12), I say that the quadrature of the figure of the one will depend on the quadrature of the figure of the other, and *vice versa*.

If, however, in place of equation (3), we took another of higher degree, the third say, we should again have two equations in place of (7) and (12); and continuing in this manner, there is no doubt that a certain definite progression of equations (7) and (12) would be obtained, so that without calculation it could be continued to infinity without much trouble. Moreover, from one given equation to any curve, all others can be expressed by a general form, and from these the most convenient can be selected.

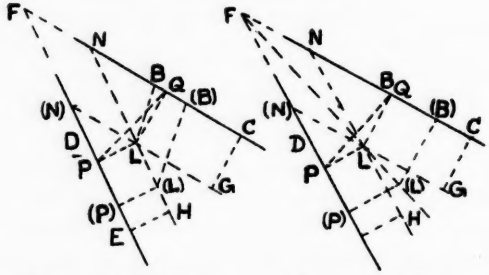
If we are given the moment of any figure about any two straight lines, and also the area of the figure, then we have its center of gravity. Also, given the center of gravity of any figure (or line) and its magnitude, then we have its moment about any line whatever. So also, given the magnitude of a figure, and its moments about any two given straight lines, we have its moment about any straight line. Hence also we can get many quadratures from a few given ones. Moreover, the moment of any figure about any straight line can be expressed by a general calculation.

The moment divided by the magnitude gives the distance of the center of gravity from the axis of libration.

[10] In view of this accurate bit of algebra, the faulty work in subsequent manuscripts seems very unaccountable.

UMI

Suppose then that there are two straight lines in a plane, given in position, and let them either be parallel or meet, when produced in F. Suppose that the moment about BC is found to be equal to ba^2, and the moment about DE is found to be ca^2. Call the area of the figure v; then the distance of the center of gravity from the straight line BC, namely CG, is equal to $\frac{ba^2}{v}$, and its distance from the straight line DE, namely EH, is equal to $\frac{ca^2}{v}$; therefore CG is to EH as b is to c, or they are in a given ratio.[11]



GERHARDT'S DIAGRAM. SUGGESTED CORRECTION.

Now suppose that the straight line EH, remaining in the plane, traverses the straight line DE, always being perpendicular to it, and that the straight line CG traverses the straight line BC, always perpendicular to it, and that the end G leaves as it were its trace, the straight line G(N), and the end H the straight HN. Then, if BC and DE meet anywhere, G(N) and HN must also meet somewhere, either within or without the angle at F. Let them meet at L; then the angle HLG is equal to the angle EFC, and PLQ (supposing that PL = EH and LQ = CG) will be the supplement of the angle EFC between the two straight lines, and will thus be a given angle. If then PQ is joined, the triangle PQL is obtained, having a given vertical angle, and the ratio of the sides forming the vertex, QL : LP, also given.

When then BL is taken, or (B)(L), of any length whatever, since the angle BLP always remains the same, and in addition we have BL to LP as (B)(L) to (L)(P), therefore also BL to (B)(L) as LP to (L)(P); and this plainly happens when FL is also propor-

[11] This proves the fundamental theorem given lower down, with regard to a pair of parallel straight lines; and he now goes on to discuss the case of non-parallel straight lines.

tional to these, that is, when a straight line passes through F, L, (L),.....

Hence, since we are not here given several regions, it follows that the locus is a straight line. Therefore, given the two moments of a figure about two straight lines that are not parallel,....., the area of the figure will be given, and also its center of gravity.[12]

Behold then the fundamental theorem on centers of gravity. If two moments of the same figure about two parallel straight lines are given, then the area of the figure is given, but not its center of gravity.

Since it is the aim of the center of gravity method to find dimensions from given moments, we have hence two general theorems:

If we are given two moments of the same figure about two straight lines, or axes of libration, that are parallel to one another, then its magnitude is given; also when the moments about three non-parallel straight lines are given. From this it is seen that a method for finding elliptic and hyperbolic curves from given quadratures of the circle and the hyperbola is evident.[13] But of this in a special note.

§ V.

The next manuscript to be considered is a continuation of the preceding, and is dated the next day. Its character is of the nature of disjointed notes, set down for further consideration.

[12] The passage in Gerhardt reads:

Datis ergo duobus momentis figurae ex duabus rectis non parallelis, dabitur figurae momentis tribus axibus librationis, qui non sint omnes paralleli inter se, dabitur figurae area, et centrum gravitatis.

For this I suggest:

Datis ergo *tribus* momentis figurae ex *tribus* rectis non parallelis, *aliter* figurae momentis tribus axibus librationis, qui non *sunt* omnes paralleli inter se....

The passage would then read:

Given three moments of a figure about three straight lines that are not parallel, in other words, the moments of the figure about three axes of libration, which are not all parallel to one another, then the area of the figure will be given and also the center of gravity.

If the alternative words are *written* down, one under the other, and not too carefully, I think the suggested corrections will appear to be reasonable.

[13] Apparently, here Leibniz is referring back to the theorem at the beginning of the article.

UMI

26 October, 1675.

Another tetragonistic analysis can be obtained by the aid of curves. Thus, let the same curve be resolved into different elements, according as the ordinates are referred to different straight lines. Hence also arise diverse plane figures, consisting of elements similar to the given curve; and since all of these are to be found from the given dimension of the curve, it follows that from the dimension of any one of the curves of this kind the rest are obtained.

In other ways it is possible to obtain curves that depend on others, if to the given curve are added the ordinates of figures of which the quadrature is either known or can be obtained from the quadrature of the given one.

Just as areas are more easily dealt with than curves, because they can be cut up and resolved in more ways, so solids are more manageable than planes and surfaces in general. Therefore, whenever we divert the method for investigating surfaces to the consideration of solids, we discover many new properties; and often we may give demonstrations for surfaces by means of solids when they are with difficulty obtained from the surfaces themselves. Tschirnhaus observed in a delightful manner that most of the proofs given by Archimedes, such as the quadrature of the parabola, and dependent theorems on the sphere, cone, and cylinder, can be reduced to sections of rectilinear solids only, and to a composition that is easily seen and readily handled.

Various ways of describing new solids.

If from a point above a plane a rigid descending straight line is moved round an area, of any shape whatever, diverse kinds of conical bodies are produced. Thus if the plane area is bounded by the circumference of a circle, a right or scalene cone is produced. Also if the figure used for the base, or the plane area, has a center—an ellipse for example—then we get an elliptic cone, which is a right cone if the given point is directly above the center, and if not it is scalene. Another conic gives another elliptic cone.

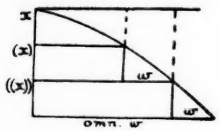
If the rigid line drawn down from the point is circular or some other curve, at one time it is so fixed to the point or pole that it has freedom to move in one way only, say round an axis, in which case it is necessary that the base should be a circle and that the fixed point or pole should be directly over the center. At another time it is necessary that the rigid line should have freedom for other motions, such as an up and down motion, or some other motion,

controlled by some straight line; and then it will always ascend or descend when necessary, so that it ever touches the given plane area by its rotation round the axis; and this is the second class of cones. A third class consists of those in which, besides the double motion of a rotation round an axis and an up and down motion, the curve alone, or the axis alone, or even both the curve and the axis, also perform other motions meanwhile, or even the point itself moves.

Here is another consideration.

The moments of the differences about a straight line perpendicular to the axis are equal to the complement of the sum of the terms; and the moments of the terms are equal to the complement of the sum of the sums, i. e.,

$$\text{omn.}\overline{xw} \sqcap \text{ult.}x,\ \overline{\text{omn.}w},, - \overline{\text{omn. omn.}w} \qquad (14)$$



Let $xw \sqcap az$, then $w \sqcap \frac{az}{x}$, and we have

$$\text{omn.}az \sqcap \text{ult.}x,\ \text{omn.}\frac{az}{x} - \text{omn.omn.}\frac{az}{x};$$

hence $$\text{omn.}\frac{az}{x} \sqcap \text{ult.}x\ \ \text{omn.}\frac{az}{x^2} - \text{omn.omn.}\frac{az}{x^2};$$

inserting this value in the preceding equation, we have

$$\text{omn.}az \sqcap \text{ult.}x^2\ \text{omn.}\frac{az}{x^2} - \text{ult.}x,\ \overline{\text{omn.omn.}\frac{az}{x^2}},$$

[14] I have given this equation, and those that immediately follow it, in facsimile, in order to bring out the necessity that drove Leibniz to simplify the notation.

We have here a very important bit of work. Arguing in the first instance from a single figure, Leibniz gives two general theorems in the form of moment theorems. The first is obvious on completing the rectangle in his diagram, and this is the one to which the given equation applies. In the other the whole, of which the two parts are the complements, is the moment of the completed rectangle; its equivalent is the equation

$$\text{omn.}xy = \text{ult.}x\ \text{omn.}y - \text{omn. omn.}y.$$

Now, although Leibniz does not give this equation, it is evident that he recognized the analogy between this and the one that is given; for he immediately accepts the relation as a general *analytical theorem that he can use without any reference to any figure whatever*, and proceeds to develop it further. This would therefore seem to be the point of departure that led to the Leibnizian calculus.

$$- \text{omn. ult.} x,\ \textit{omn.}\frac{az}{x^2} - \text{omn.omn.}\frac{az}{x^2}\ ;$$

and this can proceed in this manner indefinitely.

Again, $$\text{omn.}\frac{a}{x} \sqcap x\ \text{omn.}\ \frac{a}{x^2} - \text{omn.omn.}\frac{a}{x^2}\ ,$$

and $$\text{omn.}a \sqcap \text{ult.}x\ \text{omn.}\frac{a}{x} - \text{omn. omn.}\frac{a}{x}\ ;$$

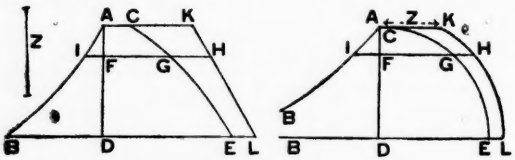
the last theorem expresses the sum of logarithms in terms of the known quadrature of the hyperbola.[15]

The numbers that represent the abscissae I usually call ordinals, because they express the order of the terms or ordinates. If to the square of any ordinate of a figure whose quadrature can be found, you add the square of a constant, the roots of the sum of the two squares will represent the curve of the quadratrix. Now if these roots of the sum of the two squares can also give an area that has a known quadrature, then also the curve can be rectified.[16]

[15] Having freed the matter from any reference to figures, he is able to take any value he pleases for the letters. He supposes that $z = 1$, and thus obtains the last pair of equations. He then considers x and w as the abscissa and ordinate of the rectangular hyperbola $xw = a$ (constant); hence omn.a/x or omn. w is the area under the hyperbola between two given ordinates, and therefore a logarithm; and thus omn. omn.a/x is the sum of logarithms, as he states.

[16] There only seem to be two possible sources for this paragraph, (1) original work on the part of Leibniz, and (2) from Barrow. For we know that Neil's methods was that of Wallis, and the method of Van Huraet used an ordinate that was proportional to the quotient of the normal by the ordinate in the original curve.

Now Barrow, in Lect. XII, § 20, has the following: "Take as you may any right-angled trapezial area (of which you have sufficient knowledge), bounded *by two parallel straight lines* AK, DL, *a straight line* AD, *and any line* KL whatever; to this let another such area be so related that when any straight line FH is drawn parallel to DL, cutting the lines AD, CE, KL in the points F, G, H, and *some determinate line* Z *is taken, the square on* FH *is equal to the squares* on FG and Z. Moreover, let the curve AIB be such that,



if the straight line GFI is produced to meet it, *the rectangle contained by* Z *and* FI *is equal to the space* AFGC; *then the rectangle contained by* Z *and the curve* AB *is equal to the space* ADLK. The method is just the same, even if the straight line AK is supposed to be infinite.

This striking resemblance, backed by the fact that there seems to be no connection between this theorem and the rest of the paper, that Leibniz gives

To describe a curve to represent a given progression.

From the square of a term of the progression, take away the square of a constant quantity; if the figure that is the quadratrix of the roots formed from the two squares is described, it will give the curve required; it does not follow that a rectifiable curve can be described.

The elements of the curve described can be expressed in many different ways. Different methods of expressing the elements of a curve may be compared with different methods of expressing a figure having similar parts with it, according as it is referred in different ways. Lastly, a solid having similar parts with a curve can thus far be expressed in many ways, and so also for a surface or figure having similar parts with the curve.

§ VI.

Three days later, Leibniz considers the possibility of being able to find the quadratrix in all cases, or when that is impossible, some curve which will serve for the quadratrix very approximately. He makes an examination of the difficulties that are likely to be met with and the means to overcome them, and he seems to be satisfied that the method can be made to do in all cases. But in the absence of an example of the method he proposes to adopt, he seems only to have been wasting his time. But this may be dismissed, for it is not here that the importance of this essay lies; it is altogether in what follows.

The rest of the essay is in the form of disjointed notes; it is just the kind of thing that any one would write *as notes while reading the works of others.* This is what I take it to be; and the works he is considering are those of

no attempt at a proof, (indeed I very much doubt whether I could have made out his meaning from the original unless I had recognized Barrow's theorem) and that Leibniz gives 1675 as the date of his reading Barrow, almost forces one to conclude that this is a note on a theorem (together with an original deduction therefrom by himself) which Leibniz has come across in a book that is lying before him, and that that book is Barrow's. Against it, we have the facts of the use of the word "quadratrix," not in the sense that Barrow uses it, namely as a special curve connected with the circle; that the quadratrix is one of the special curves that Barrow considers in the five examples he gives of the Differential Triangle method; and that another example of this method is the differentiation of a trigonometrical function which seems to be unknown to Leibniz.

Descartes, Sluse, Gregory St. Vincent, James Gregory and Barrow. Descartes he has already dismissed as impracticable in the manuscript of January, 1675; but there are indications that the former's method has still some influence. An incidental remark leads to the consideration of the *ductus* of Gregory St. Vincent; but these too are soon cast aside, truly because Leibniz does not quite grasp the exact meaning of Gregory. He then either remembers what he has seen in Barrow or refers to it again, for the next thing he gives is some work in connection with which he draws the characteristic triangle, *which is here for the first time, as far as* these manuscripts *go, the Barrow form and not the Pascal form.* He immediately obtains something important, namely,

$$\frac{\overline{\text{omn. } l^2}}{2} = \text{omn. } \overline{\overline{\text{omn. } l}\,\frac{l}{a}}.$$

Noting that, in modern notation, l is dy, and a is dx, and also, since a is also supposed to be unity, that the final summation on the right-hand side is performed by "applying the successive values to the axis of x, while the summation denoted by omn.l is a straightforward summation, it follows that the equivalent of the result obtained by Leibniz is $\frac{1}{2}y^2 = \int y \frac{dy}{dx} dx$.

However, in attempting to put this theorem into words as a general theorem he makes an error; he quotes $\overline{\text{omn.}l^2}$ as the "sum of the squares" instead of the "square of the final y." This I think is simply a slip on the part of Leibniz, and not, as suggested by Gerhardt and Weissenborn, an indication that Leibniz confused $\overline{\text{omn.}l^2}$ with $\overline{\text{omn.}l}^2$, and considered them as equivalent. Neither of these authorities appears to have noticed the fact that when Leibniz has invented the sign $\int$ (which he immediately proceeds to do) he carefully makes the distinction between the

equivalents to the square of a sum and the sum of the squares. Thus we find that his equation is written as

$$\int \frac{\overline{l^2}}{2} = \int \overline{\int l} \frac{l}{a}, \qquad \text{(note the vinculum)}$$

while later in the essay we have $\int l^3$ to stand for the sum of the cubes. Further, apart from this. I do not think that any one can impute such confusion of ideas to Leibniz, if it is noted that so far this is not the differential calculus, but the calculus of differences, i. e., l is still a very small but finite line and not an infinitesimal; for in § IV, Leibniz had squared a trinomial successfully, and must have known that the sum of the squares could not be equal to the square of the sum. Both these above-named authorities seem to find some difficulty over the introduction of the letter a, apparently haphazard. This difficulty becomes non-existent, if it is remembered that a is taken to be unity, and the remarks made about dimensions by Leibniz are carefully considered; it will then be found that the a is introduced to keep the equations homogeneous! Weissenborn also remarks that Leibniz jots down the integral of x^2 without giving a proof, and appears to be in doubt how he reached it. If this is so, it confirms the opinion that I have already formed, namely, that neither Gerhardt nor Weissenborn tried to get to the bottom of these manuscripts, being content with simply "skimming the cream."

I suggest that Barrow, Gregory St. Vincent, and even Sluse, now join Descartes on the shelf or the floor, and that the rest of the essay is all Leibniz. He writes the two equations he has found, the equivalents to two theorems obtained geometrically, notes the fact that these are true for infinitely small differences (without, however, mentioning that they are *only* true in such a case), discards diagrams, and proceeds analytically; that is, the y's are successive values of some function of x, where the values

of x are in arithmetical progression; hence, substituting x for l in the equation

$$\text{omn.}xl = \text{omn.}l - \text{omn.omn.}l,$$

and remembering that $\text{omn.}x = x^2/2$, as he has proved, we have

$$\text{omn.}\, x^2 = x\,\frac{x^2}{2} - \text{omn.}\frac{x^2}{2}, \text{ or } \text{omn.}\, x^2 = \frac{x^3}{3}.$$

Again, below he gives $\int \frac{x^3}{3} = \frac{x^4}{4}$ correctly (although there is an obvious slip or, as I think, a misprint of l for x); this could have been obtained in the same way.

$$\text{omn.}\, x^3 = x\frac{x^3}{3} - \text{omn.}\frac{x^3}{3}, \text{ or } \text{omn.}\, x^3 = \frac{x^4}{4}.$$

Similarly, Leibniz could have gone on indefinitely, and thus obtained the integrals of all the powers of x. But his brain is too active; as Weissenborn says, his soul is in the throes of creation. He merely alludes in passing to the inverse operation to $\int$ as being represented by d, which he for some reason writes in the denominator (probably erroneously because he has noted that $\int$ increases the dimensions); and then he harks back to the opening idea of the essay, the obtaining of the quadratrix by means of transformation of equations, an idea truly as hopeless as the method of Descartes which he has discarded. Nevertheless, even then he obtains something remarkable, nothing more or less than the inverse of the differentiation of a product. This fundamental theorem is obtained geometrically; the proof of the little theorem on which the final result is founded is not given, neither is there a diagram. It cannot therefore be supposed but that Leibniz is working from a diagram already drawn, and I suggest he was referring to one of those theorems, with which he had filled "hundreds of pages" between 1673 and 1675. The

proof follows quite easily by the use of the characteristic triangle, and is given in a footnote. This theorem is not in Barrow, nor can I remember seeing it in Cavalieri; I have not yet been able to procure a Gregory St. Vincent; it may be in James Gregory.

The benefits of this discovery are lost as before, for Leibniz once more alludes to the transformation of equations for the purpose of obtaining the quadratrix.

Summing the whole essay, we can say that in it is the beginning of the Leibnizian *analytical calculus.*

29 October, 1675.

Analyseos Tetragonisticae pars secunda.

(Second part of analytical quadrature.)

I think that now at last we can give a method, by which the analytical quadratrix may be found for any analytical figure, whenever that is possible; and, when it can not be done, it will yet always be possible that an analytical figure may be described, which will act as the quadratrix as nearly as is required. This is how I look at it:

Suppose the equation of the curve, of which the quadratrix is required, is given, and that the unknowns in it are x and v. Let the equation to the curve required be[17]

$$v = b + cx + dy + ex^2 + fy^2 + gyx + hy^3 + lx^3 + mxyy + yxx + \text{etc.}; \quad \ldots \quad \text{(i)}$$

let it be set in order for tangents, as follows:

$$-dy - 2fy^2 - gyx - 3hy^3 - 2mxy^2 - mx^2y - \text{etc.}$$
$$= ct + 2ext + gyt + 3lx^2t + my^2t + 2yxt + \text{etc.} \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

[17] This is either a misprint, v instead of O, or else Leibniz is in error. For Slusius's method there must be only two variables in the equation. In the *Phil. Trans.* for 1672 (No. 90), Sluse gives his method thus:

If $y^5 + by^4 = 2qqv^3 - yyv^3$, then the equation must be written $y^5 + by^4 + yy^3 = 2qqv^3 - yyv^3$; then multiply each term on the left-hand side by the number of y's in the term, and substitute t in place of one y in each; similarly multiply each term on the left-hand side by the exponent of v; the equation obtained will give the value of t.

The use of the letters v and y is to be noted in connection with Leibniz's use of the same letters; it does not seem at all necessary that Leibniz should have seen Newton's work, with this ready to the former's hand, as a member of the Royal Society. I suggest that Sluse obtained his rule by the use of *a* and *e*, as given in Barrow. Can Barrow's words *usitatum a nobis* (in the midst of a passage written in the first person singular) have meant that the method was common property to himself *and several other mathematicians that were contemporary with him*? This would explain a great deal.

Now, $t/y=a/v$; hence, if from the equation $t/y=a/v$, we eliminate t and y by the help of equations (i) and (ii), that equation should be produced which represents the figure that has to be quadratured; and by comparing the terms of the equation thus obtained with the given equation, unless indeed there is no possibility of comparing them, we shall obtain the quadrature.

But if an impossibility arises, it is then known that the given analytical figure has no analytical quadratrix. But it is quite clear that if we add to it such as will change it almost imperceptibly, then a quadrible figure may be obtained, since this plainly produces another equation. However, as an impossible case may arise, we must consider the difficulties.

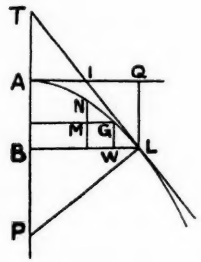
Say that the equation that is obtained is of infinite prolixity, while the given one is finite. My answer is, that in comparing the one with the other it will be seen how far at most the powers of the unknowns in the indefinite equation can go. The retort may be made, that it may happen that the indefinite equation obtained may have more terms than the finite equation that is given and yet may be reduced to it, for it may be divided by something else that is either finite or indefinite. This difficulty hindered me for a long time a year ago, but now I see that we should not be stopped by it. For it may happen that from a certain determinate figure (whose equation is not divisible by a rational) by the method of tangents there may arise an ambiguous figure; for it is impossible to say that, for *any* figure, there shall be only one tangent at any one point. Hence the produced equation can neither be divided by a finite nor by an indefinite quantity; for in truth indefinite figures, or those whose ordinates are represented by an infinite equation, have sometimes these very ordinates finite, and these ought to satisfy the equation. Notwithstanding that, I foresee another difficulty; for indeed it seems that sometimes it may happen that all the roots of the equation will not serve for the solution of the problem. Yet, to tell the truth, I believe they will do so.

Now here is a difficulty that really is great. It may happen that a finite equation may also be expressed as an indefinite one, so that the equation obtained may really be the same as the given equation although it does not appear to be. For example,

$$y^2=x/(1+x)=x-x^2+x^3-x^4+x^5-x^6+\text{etc.};$$

and in the same way others can be formed by various compositions and divisions. This I confess is truly a difficult point, but it can be

answered thus: If a figure has an analytical quadratrix of any sort, in all cases it may be assumed to be an indefinite one; and then it will not in all cases give an indefinite, but sometimes a finite, equation that is equivalent to the given equation. In the same way, it is certain that the quadratrix of a given curve as it is usually investigated, whenever there is one, will also be determined; and that too given uniquely and not ambiguously, so that any that differs from it, differs only in name. There is still one difficulty left; it seems impossible to determine which is the end or first term of the indefinite equation that is obtained; for it may happen that the terms of lower degree are cut out, and then it is divisible by y or x or yx or powers of these; nor do I see that there is anything to prevent this. There is the same difficulty whether you start from the lowest or the highest degree in the equation assumed to begin with as indefinite. Suppose then that in the equation obtained this



division is possible, then it is necessary that the constant term should be absent, and also all those terms in which x alone or, if you like, all the terms in which y alone is absent; and if we examine this continuously we may light upon an impossibility.

In this general calculus then, we may take it as certain that this difficulty is solved, and that such a division after the calculation can never happen; or if it is possible for it to happen, then the terms will go out, one after the other, so that the equation can be depressed and the comparison be made; and then it is to be seen whether this difficulty cannot be overcome in general, and the comparison proceed as we proceed with the elimination. Perhaps if the figure to be quadratured is reduced beforehand to its simplest equation possible, impossibilities will the more readily be detected. For then presumably the quadratrix must become more simplified. In addition we have another source of assistance; for various cal-

culations leading to the same thing, though obviously differing from one another, can be contrived, from which equations are comparable.

Let BL = y, WL = l, BP = p, TB = t, GW = a, then y = omn.l.

Incidentally I may remark that there are composite numbers that cannot be added or subtracted from one another by parts, namely those denominated by powers, or by sub-powers or surds. There are also other denominate numbers which cannot be multiplied together by parts, such as numbers representing sums; for instance, omn.l cannot be multiplied by omn.p, nor can we have y^3 = 2omn. omn.pl. However, as such a multiplication may be imagined to occur under certain conditions, we must consider it as follows:

We require the space that represents the product of all the p's into all the l's; we cannot make use of the ductions of Gregory St. Vincent, where figures are multiplied by figures, for by this method one ordinate is not multiplied by all the others, but one into one. You may say that if one ordinate is multiplied by all the rest it will produce a sursolid space, namely, the sum of an infinite number of solids. For this difficulty I have found a remedy that is really admirable. Let every l be represented by an infinitely short straight line WL, that is, we want the quadratrix line representing omn. l; well, the line BL = omn. l; and if this is multiplied by every p, each represented by a plane figure, then a solid is produced. If all the l's are straight lines and all the p's are curves, a curved surface is produced by a duction of the same sort. But these things are all old; now, here is something new.

If upon WL, MG, or every single l, is superimposed the same curve representing all the p's, where the curve p is originally all in the same plane and is carried along the curve AGL while its plane always moves parallel to itself, then what we require will be obtained. In place of a curve a plane may be carried along the curve in the same manner, and a solid will be obtained, whereas by the former method it was a curvilinear surface; and both for the surface and for the solid the section always remains the same. It remains to be seen whether a number of analytical surfaces cannot be ascertained, as in the case of analytical lines; but this is mentioned only incidentally.

N. B. The curvilinear surface formed by the motion of a curve parallel to itself along the curve will be equal to the cylinder

of the curve under BL, the sum of all the l's but this is also mentioned incidentally.

To resume, $\frac{l}{a} = \frac{p}{\text{omn. } l} = y$, therefore $p = \frac{\overline{\text{omn. } l}}{a} l$. Hence, omn. $y \frac{l}{a}$ does not mean the same thing as omn.y into omn.l, nor yet y into omn.l; for, since $p = \frac{y}{a} l$ or $\frac{\overline{\text{omn. } l}}{a} l$, it means the same thing as omn.l multiplied by that one l that corresponds with a certain p; hence, omn.p = omn.$\frac{\overline{\text{omn. } l}}{\text{a}} l$. Now I have otherwise proved omn.$p = \frac{y^2}{2}$, i. e., $= \frac{\overline{\text{omn. } l^2}}{2}$; therefore we have a theorem that to me seems admirable, and one that will be of great service to this new calculus, namely,

$$\frac{\overline{\text{omn. } l^2}}{2} = \text{omn. } \overline{\overline{\text{omn. } l} \frac{l}{a}}, \text{ whatever } l \text{ may be;}$$

that is, if all the l's are multiplied by their last, and so on as often as it can be done, the sum of all these products will be equal to half the sum of the squares, of which the sides are the sum of the l's or all the l's. This is a very fine theorem, and one that is not at all obvious.

Another theorem of the same kind is:

$$\text{omn.} xl = x \text{ omn.} l - \text{omn.omn.} l,$$

where l is taken to be a term of a progression, and x is the number which expresses the position or order of the l corresponding to it; or x is the ordinal number and l is the ordered thing.

N. B. In these calculations a law governing things of the same kind can be noted; for, if omn. is prefixed to a number or ratio, or to something indefinitely small, then a line is produced, also if to a line, then a surface, or if to a surface, then a solid; and so on to infinity for higher dimensions.

It will be useful to write $\int$ for omn., so that

$$\int l = \text{omn.} l, \text{ or the sum of the } l\text{'s.}$$

Thus, $\int \frac{\overline{l^2}}{2} = \int \overline{\int \overline{l} \frac{l}{a}}$, and $\int \overline{xl} = x \int \overline{l} - \int \int l$.

From this it will appear that a law of things of the same kind

should always be noted, as it is useful in obviating errors of calculation.

N. B. If $\int l$ is given analytically, then l is also given; therefore if $\int \int l$ is given, so also is l; but if l is given, $\int l$ is not given as well. In all cases $\int x = x^2/2$.

N. B. All these theorems are true for series in which the differences of the terms bear to the terms themselves a ratio that is less than any assignable quantity.

$$\int x^2 = \frac{x^3}{3}$$

Now note that if the terms are affected, the sum is also affected in the same way, such being a general rule; for example, $\int \frac{a}{b} l = \frac{a}{b} \times \int l$, that is to say, if $\frac{a}{b}$ is a constant term, it is to be multiplied by the maximum ordinal; but if it is not a constant term, then it is impossible to deal with it, unless it can be reduced to terms in l, or whenever it can be reduced to a common quantity, such as an ordinal.

N. B. As often as in the tetragonistic equation, only one letter, say l, varies, it can be considered to be a constant term, and $\int l$ will equal x. Also on this fundamental there depends the theorem:

$$\int \frac{l^2}{2} = \int \overline{\int l}\, l, \text{ that is, } \frac{x^2}{2} = \int x.$$

Hence, in the same way we can immediately solve innumerable things like this; thus, we require to know what e is, where

$$\int \frac{c}{a} \overline{\int l + ba^2} + \int l^3 \overset{(18)}{+} \int l^3 = ea^3;$$

we have

$$a^3 e = \frac{cx^3}{3} + ba^2 x + \frac{x^4}{4} + xa^3.$$

For indeed $\int l^3 = x$, because l is supposed to be equal[19] to a for the purpose of the calculation; $\int \frac{l}{a} = x$.

[18] There is evidently a slip here; l should be x.

[19] This is an instance of the care which Leibniz takes; in the work above l has been the difference for y, and a the difference for x; he is now integrating an algebraical expression, and not considering a figure at all; hence $l = a$, and a is equal to unity, and therefore $\int l^3 = l^3 x = a^3 x = x$! Thus what is generally considered to be a muddle turns out to be quite correct. The muddle is not with Leibniz, it is with the transcriber. It is certain that these manuscripts want careful republishing from the originals; won't some millionaire pay to have them reproduced photographically in an *edition de luxe*?

Also $\int c \int \overline{l^2} = \frac{cx^3}{3}$, that is $= \frac{c \int \overline{l^3}}{3a^3}$, $\int ba^2 = \int l\, ba$.

Also it is understood that a is unity. These are sufficiently new and notable, since they will lead to a new calculus.

I propose to return to former considerations.

Given l, and its relation to x, to find $\int l$.

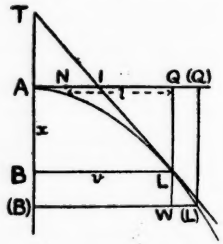
This is to be obtained from the contrary calculus, that is to say, suppose that $\int l = ya$. Let $l = ya/d$; then just as $\int$ will increase, so d will diminish the dimensions. But $\int$ means a sum, and d a difference. From the given y, we can always find y/d or l, that is, the difference of the y's. Hence one equation may be transformed into the other; just as from the equation $\int c \int \overline{l^2} = \frac{c \int \overline{l^3}}{3a^3}$, we can obtain the equation $c \int \overline{l^2} = \frac{c \int \overline{l^3}}{3a^3 d}$.

N. B. $\int \frac{x^3}{b} + \int \frac{x^2 a}{e} = \int \frac{x^3}{b} + \frac{x^2 a}{e}$. And in the same manner, $\frac{x^3}{db} + \frac{x^2 a}{de} = \frac{\frac{x^3}{b} + \frac{x^2 a}{e}}{d}$.

But to return to what has been done above. We can investigate $\int l$ in two ways; one, by summing y and seeking $ya/d = 1$; the other, by summing $z^2/2a = y$, or by summing $\sqrt{2ay} = z$, and then $z^2/t = p = l = ya/d$. Hence, if in an indefinite equation, we eliminate y by substituting in its place $z^2/2a$, and investigate the t of this new equation which is indefinite like the first, and then by the help of the value $z^2/t = l$, and after that by the help of the new value of t, eliminate z from the indefinite equation containing z and t, there will remain out of the (three) letters x, z, t, l, the letter l alone; and again we ought to get an equation which should be the same not only as the given one, but also the same as the one that was obtained a little while ago. Hence, since we have two indefinite equations, containing not only the principle quantities, but also arbitrary ones, yet not altogether unlike the former; and these ought to be identical; it will appear to show whether certain terms cannot be eliminated, whether it is not possible that a comparison should be made, and other things of the sort; and, what is really the most important thing, which terms are really the greatest and the least, or the number of terms of the equation.

Moreover, since in the similar triangles TBL, GWL, LBP, no

mention has yet been made of the abscissa x or of the fixed point A, let us then suppose that through the fixed point A there is drawn an unlimited straight line AIQ, parallel to LB, meeting the tangent LT in I; and let AQ = BL; bisect AI in N; then I say that the sum of every QN will always be equal to the triangle ABL, as can easily be shown by what I have said in another place.[20]



These considerations give once more a fresh fundamental theorem for the calculus. For $xv/2 = y$, where we suppose that $BL = v$ and $QN = l$, and $y = \int l$;

but $\frac{AI}{v} = \frac{t-x}{t}$, therefore $AI = \frac{t-x}{t} v$,

and $QI = v - AI = v\frac{tt}{t}v + \frac{xv}{2t}$, i. e. $QI = \frac{xv}{t}$,

$$\therefore QN = QI + \frac{AI}{2} = \frac{xv}{t} + \frac{v}{2} - \frac{xv}{2t} = \frac{xv + tv}{2t} = l. \quad (21)$$

Now, by the help of the equation $(xv+tv)/2t = l$, and of the former equation $y = xv/2$, and taking once more the first indefinite or general equation as a third, and eliminating first of all y, then t by means of the value found for the ratio of t to x from the indefinite equation containing x and v, and lastly v by the help of the equation $(xv+tv)/2t = l$, in which the principal quantities x and l alone remain, as before; and this again should be identical with the given equation.

Thus we have found three equations obtained in different ways, which should all be identical with one another and with the given equation; and these three are not only identical but should also

[20] Since the triangles QLI, WL(L) are similar, QI.B(B) = QL.Q(Q), hence omn.QI (applied to AB) = omn. QL (to AQ) = figure AQLA, hence omn.(QI + QA) = rect. ABLQ = 2△ABL.

[21] Since l is the difference for y, therefore $2l$ is the difference for xv; this is shown to be $(xv + tv)/t$ or $x(v/t) + v$; and this is the equivalent to (since $v/t = dv/dx = dv$)

$$d(xv) = xdv + v = xdv + vdx.$$

consist of the same letters and signs; and whether this is possible, will immediately appear on being worked out analytically.

§ VII.

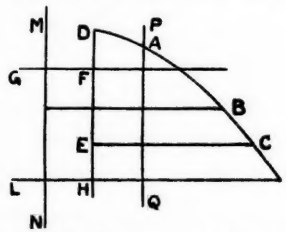
The next manuscript is a further continuation of the preceding, written two days later. In this Leibniz returns to the idea that he has found so prolific, namely, the moments of a figure. It is to be observed that he speaks of the method of breaking up an area into segments as something that he has already worked out; this will be remarked upon in a note on a later manuscript, where it will help to clear up a small difficulty. The accuracy of the rather involved algebraical work is also a point to be noticed.

1 November, 1675.

Analyseos Tetragonisticae pars tertia.

(Third part of Analytical Quadrature.)

It was some time ago that I observed that, being given the moment of a curve ABC, or of a curvilinear figure DABCE, about two straight lines parallel to one another, such as GF, LH (or MN,



PQ), then the area of the figure could be obtained; because the two moments differed from one another by the cylinder of the figure, where the altitude was the distance between the parallels.

Now, this is true of every progression, whether of numbers or of lines; that is, even if we do not use curvilinear figures but ordinated polygons; in other words, where the differences between the terms are not infinitely small. Suppose we have any such ordinated quantity z, and let the ordinal number be x, then

$$b\,\overline{\text{omn.}z} \sqcap \pm \text{omn.}\overline{zx} \mp \text{omn.}\overline{zx+b}$$

and this is evident by the calculus alone.

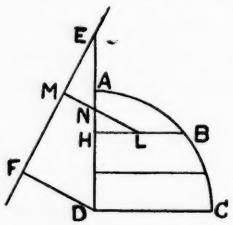
By the help of this rule, the sums of terms of an arithmetical

progression refolded reciprocally;[22] and this multiplication takes place when it is required to find the moment of the ordinates about a straight line perpendicular to the axis. But if the moment about any other straight line is required, there is the following general rule:

From the center of gravity of each of the quantities of which the moment is required, a perpendicular is drawn to the axis of libration; then the sum of the rectangles contained by the distances or perpendiculars and the quantities will be equal to the moment about the given straight line.

Hence, if the given straight line is the axis of equilibrium, it immediately follows that the moment of the figure about the axis is equal to the sum of the half-squares. Also when it is parallel to that, it will differ from the foregoing by a known quantity.

Now, let us take another straight line: for the circle for instance, let ABCD be a quadrant, vertex B, and center D; let another straight line be given, that is to say, let the prependicular DF be given and



also EF where it meets the diameter, and thus also DE; let HB be the general ordinate to the circle, and L its middle point; let LM be drawn perpendicular to EF.

Then it is clear that the triangles EFD, EMN (where N is the intersection of ML and AD), and LHN are similar.

Let AD$=x$, then $\mathrm{HL} = \frac{y}{2} = \frac{\sqrt{a^2 - x^2}}{2}$. But, on account of the similar triangles, $\frac{\mathrm{NH}}{\mathrm{HL}} = \frac{\mathrm{DF}(=d)}{\mathrm{FE}(=f)}$;

therefore

$$\mathrm{NH} = \frac{d}{2f}\sqrt{a^2 - x^2} = \frac{yd}{2f}.$$

[22] The meaning of this is probably a series such as that considered by Wallis. If a, $a+d$, $a+2d$, etc. is the arithmetical progression, and l, $l-d$, $l-2d$, etc. is the series reversed, then the series refolded reciprocally is al, $(a+d)(l-d)$, $(a+2d)(l-2d)$, etc. It may however mean the sum of the squares of the arithmetical progression. But the point is not very important.

Hence, $EN = DE(=e) - HD(=x) - NH\left(=\frac{yd}{2f}\right) = e - x - \frac{yd}{2f}$.

Now $NL = \sqrt{NH^2 + HL^2} = \sqrt{\frac{d^2}{4f^2}y^2 + \frac{y^2}{4}} = \frac{y}{2}\sqrt{\frac{d^2}{f^2}+1}$;

and $\frac{MN}{EN} = \frac{NH}{HL}$, or $MN = \frac{NH \cdot EN}{HL}$; thus we have

$$MN = \frac{dy}{2f}\frac{e - x - \frac{dy}{2f}}{\frac{y}{2}\sqrt{\frac{d^2}{f^2}+1}} = \frac{d}{f\sqrt{\frac{d^2}{f^2}+1}}\overline{e - x - \frac{dy}{2f}};$$

and $$ML = MN + NL = \frac{d}{f\sqrt{\frac{d^2}{f^2}+1}}\overline{e - x - \frac{dy}{2f}} + \frac{y}{2}\sqrt{\frac{d^2}{f^2}+1};$$

hence, since $e = \sqrt{f^2 - d^2}$, we have [23]

$$ML = \frac{d\overline{\sqrt{f^2-d^2} - x - \frac{d}{2f}y} + \frac{d^2+f^2}{2f}y}{\sqrt{d^2+f^2}} = \frac{d\overline{\sqrt{f^2-d^2} - x} + \frac{fy}{2}}{\sqrt{d^2+f^2}}$$

and this calculation is general for any curve, so long as x is always taken as the abscissa and y as the ordinate.

Therefore the rectangle contained by ML and HB $(=y)$, or the **moment of each ordinate** taken with regard to the straight line EF, **or *wa*, will be equal to**

$$\frac{d\overline{\sqrt{f^2-d^2}y - xy} + \frac{f}{2}y^2}{\sqrt{f^2+d^2}}$$

Hence, omn.w will be obtained from the known values of omn.x, omn.xy, and omn.y^2; also, if any three of these four are given, the fourth is also known.

Now, omn.xy will be equal to the moment of the figure about the vertex, omn. y^2 will be equal to the moment of the figure about the axis; hence, given three moments of the figure, that is to say, the moments about two straight lines at right angles and any third, the area is given.

This theorem, however, is less general than the one that was given before, in the first part of this essay, where it does not matter

[23] The accuracy of the algebra is noteworthy in comparison with the inaccuracies that occur later. There is however a slip: $e^2 = f^2 + d^2$ and not $f^2 - d^2$; this must be a slip and not a misprint, because it persists throughout. It should be noted that the figure given by Gerhardt is careless in that LM is made to pass through A.

what the angle between the straight lines may be, if only we are given three moments; but it is always understood that they are in the same plane. (Meanwhile, however, this theorem will suffice for the curve of the primary hyperbola; for, if f is infinite, or if FE and ED are parallel, $dy + y^2/2 = wa$, as has already been proved.)

It is to be observed that by other calculation the area of a quantity, whose center of gravity lies in a given plane (even though the whole quantity does not), can be found from three given moments about three straight lines in that plane. From this it is to be seen whether the results obtained, when compared with one another, will not produce something new.

If instead of the moment of a figure we require the moment of all the arcs BP, PC, etc., the perpendiculars are to be drawn from the points B, P, C, etc. only, to the straight line; for it will make no difference whether they are drawn from the end or from the middle of BP, for instance, for the difference between two such perpendiculars is infinitely small. Hence, calling the element of the curve z, the moment of the curve about the straight line EF is

$$\frac{d\sqrt{f^2 - d^2}z - dxz + fyz}{\sqrt{d^2 + f^2}}$$

Most of the theorems of the geometry of indivisibles which are to be found in the works of Cavalieri, Vincent, Wallis, Gregory and Barrow, are immediately evident from the calculus; as, for instance, that the perpendiculars to the axis are equal to the surface or moment of the curve about the axis, for you find that a perpendicular is equal to the rectangle contained by an element of the curve and the ordinate. Therefore I do not set any value on such theorems, or on those about applications of intercepts on the axis (intercepted between the tangents and the ordinates) to the base. Such theorems bring forth nothing new, except maybe they afford formulas for the calculus.

But my theorem about the dimensions of the segments does bring out a new thing, because the space whose dimension is sought is broken up in a different way, that is to say, not only into ordinates but into triangles. Also perhaps the Centrobaric method yields something new. Maybe an easy method can be obtained, by which without diagrams those things which depend on a figure can be derived by calculus. Gregory's theorem, on ductions of two

parabolas,[24] one under the other, equal to a cylinder, is immediately evident by calculus; for the ordinate of a circle $y=\sqrt{a^2-x^2}$, that is, the product of $\sqrt{a+x}$ and $\sqrt{a-x}$; and in the same way, $\sqrt{2av-v^2}=y$, which gives $y=\sqrt{v}$ into $\sqrt{2a-v}$; and these come to the same thing.

If the same ordinate y is multiplied by some quantity z, and afterward by the same $z \pm$ some known or constant number b, the difference between the sums produced will be equal to the cylinder of the figure; so that

$$zy,,-zy+by \sqcap by.$$

Although this is evident in general by itself, yet applications of it are not always evident. For instance, let

$$y=\frac{x^2}{ax-b^2}=\frac{x^2}{\sqrt{ax}+b,\ \sqrt{ax}-b};$$

then, multiplying by $\sqrt{ax}+b$, we have $\frac{x^2}{\sqrt{ax}-b}$; ..... (A)

and, multiplying by $\sqrt{ax}-b$, we have $\frac{x^2}{\sqrt{ax}+b}$; ..... (B)

but, since instead of $\frac{ax^2}{ax-b^2}$, we can have $x+\frac{b^2x}{ax-b^2}$,

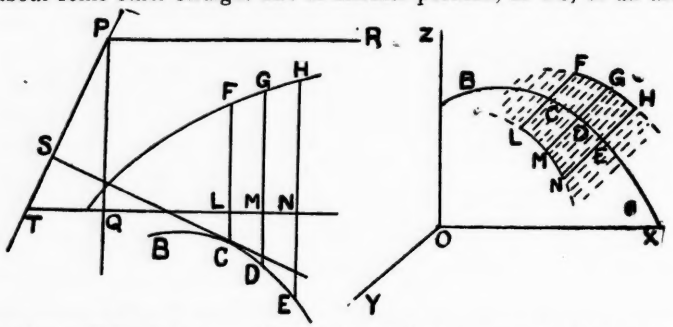
which depends on the quadrature of the hyperbola; and thus if one of the two things, (A) or (B), is given, then the other is also known, supposing that the quadrature of the hyperbola is known.

Suppose that at the points C, D, E of a curve situated in any plane there are imposed, perpendicular to the plane, the ordinates of another curve FGH (not necessarily of the same constitution), in such a manner that the middle point of each of these ordinates lies in the plane; then it is evident that LG, MD, NE, multiplied by FL, GM, HN, (that is, the lines imposed at C, D, E of the curve BCDE) or the rectangles FLG, GMD, HNE, or the duction of these two planes into one another, will be equal to the moment of every LC, MD, NE, etc. Hence, if PR is another axis, and the interval between it and QL is the straight line PQ, the moment

[24] Such theorems are also considered in Wallis, where it is shown that the products for two equal parabolas are the squares on the ordinates of a semicircle; the axes of the parabolas being coincident, but set in opposite sense.

about PR differs from that about QL by the cylinder whose base is LC, MD, etc., and whose height is PQ.[25]

But, if the moment about the straight line PQ, and also that about some other straight line in another position, as TS, of all the



ordinates LF of the same figure, imposed at the points C, then we shall have the cylinder corresponding to all the LF's, as I will now prove.

If we call QL, x, and CL, y, then $TC = \frac{f}{a}x + \frac{g}{a}y + h$; and this multiplied by z, where FL or $MG = z$, will give

$$\frac{f}{a}xz + \frac{g}{a}yz + hz.$$

Now xz is given, being the supposed moment about PQ, which is the same whether the z's are placed where they were in the lines LF, MG, etc., or at the points C, D, E. Also yz is given, either as the rectangle FLC or as the duction, by hypothesis. Hence, if in addition there is given one moment of the ordinates imposed upon

[25] This is obviously wrong; the base of the cylinder is the area made up of FL, GM, HN, etc. The whole of this last passage proved to be difficult to make out; Leibniz has not completed his figure, by showing the surface formed by placing the ordinates FL, GM, HN with their middle points at C, D, E, and the ordinates themselves perpendicular to the plane of the curve BCDE, which figure I have added on the right-hand side of Leibniz's figure. Even when this is given, there is another difficulty added because as given by Gerhardt, CS is the tangent at D instead of the proper line, namely, the perpendicular from C to TS; in addition through a misprint, this line is afterward referred to as TC. Lastly, "the rectangle FLG" is a misprint for FLC, which with Leibniz stands for FL.LC; this notation for a rectangle is, as far as I can remember, used by Wallis and Cavalieri.

When all these errors are revised, what at first sight seemed to be rather a muddle turns out to be an exceedingly neat idea in connection with the moments of a figure, and their use to find an area, although mostly impracticable.

Note. The values f, g, a, h, are the lengths of TQ, QP, PT, and the perpendicular from Q on PT.

the curve at the points C, D, E, and this is taken to be equal to $\frac{f}{a}xz + \frac{g}{a}yz + hz$, then we have hz or the cylinder required.

Hence, the curve BCDE is to be chosen such that the ordinates of the given curve can be multiplied by different ordinates of the former, drawn either to the axis QL or to the axis TS, with some advantage of simplicity; and the curves that are suitable for this are those that have several suitable axes, such as the circular or primary hyperbola, which has a pair of asymptotes, or an axis and a conjugate axis.

§ VIII.

Much comment has been made on the fact that the date of the next manuscript was originally "11 November 1675"; that the 5 had been altered to a 3, the ink being of a darker shade; and that it is almost certain that this alteration in date was made for some ulterior motive by Leibniz himself. Hence, if he was capable of falsifying a date in one particular case, then he is not to be trusted in others,. . . ., and so on. Instead of trying to explain away this alteration, let us try to find an explanation as to the reason of its having been made by Leibniz; I offer the following as at least feasible.

The essay starts with the words, "*Jam superiore anno mihi proposueram questionem,. . . .*" I suppose that by this Leibniz intended: "A year or two ago, I set myself the question,. . . . ." This conforms with what follows; the theorem that he sets down is one such as those that were suggested to him by Huygens, and further theorems that came to him as deductions during his first intercourse with Huygens. Years later, I therefore suggest, Leibniz refers to this manuscript, reads his own Latin, *superiore anno,* as "in the above year," gets no further, recognizes the theorem by its figure as one of the Huygens-time batch, and says to himself "1675? No, that's wrong, should be 1673,"

and proceeds to alter it to what he remembers was the date for the first consideration of the theorem.

N. B. Gerhardt himself has remarked on the darker tint of the ink used in the alteration; hence my argument, made at a later date.

The date 1675 is incontestable; for this composition is quite glaringly a development of the work that has been so efficiently started in that of November 1, 1675. Progress is still delayed by the idea that has obsessed Leibniz up till now, that of the transformation of equations, so as to be able to eliminate more unknowns than the original number of his equations warrant. He sets himself the problem: "To determine the curve in which the distance between the vertex and the foot of the normal is reciprocally proportional to the ordinate," i. e., the solution of the equation $x + y\,dy/dx = a^2/y$, in modern notation. This is a very unlucky choice for him: for I have it on the authority of Prof. A. R. Forsyth that this is incapable of solution in ordinary functions or even by a series in which the law of the series is easily and simply expressible—at least he confesses that he is unable to obtain such a solution, which I take it comes to the same thing.

Leibniz professes to have found the solution and gives $(y^2 + x^2)(a^2 - yx) = 2y^2 \log y$; and unfortunately this false success but enhances the value in his eyes of the method mentioned above. But from the equation given as the solution we may draw an incontestable conclusion; for in a previous problem Leibniz verifies his solution by the method of tangents, i. e., by differentiation, although the method does not as yet convey that idea to him; but he does not verify the solution in this case, *because he is unable at this date to differentiate the product* $y^2 \log y$.

The introduction of dx instead of x/d marks a further advance, more important perhaps than the use of $\int y\,dy$;

for he still writes $\int x$, considering dx to be constant and equal to unity. He is beginning to grasp the infinitesimal nature of his calculus, and that infinitesimals are not to be neglected because of their intrinsic smallness, but because of their smallness *with respect to other quantities* which come into the same equations and are finite; but he is far from being certain about it as yet, as is evidenced by the discussion as to whether $d(v/\psi) = dv/d\psi$ or not. However, the whole manuscript marks a distinct advance on anything that has gone before. From now on he probably discards geometry, and only refers to Descartes, Gregory and Barrow for examples to show how much superior is his method to theirs. I put his final reading of Barrow down to the interval between the date of this manuscript, 11 November, 1675, and November, 1676; it is at this time that he inserts his sign of integration in the margins of the theorems. The next person that examines the originals of these manuscripts (I am convinced that this is very necessary), should carefully see whether the ink used for the note *"novi dudum"* (which I have mentioned) is the same as that used for the sign of integration; also the other books that were used by Leibniz in his self-education should be searchingly scrutinized for clues.

The last remark I have to make is one of astonishment at the errors in the algebraical work which brings this essay to a close, and to a less degree throughout the essay; for we have seen the accuracy to which Leibniz has attained in a previous manuscript; of course, a great deal of erroneous work can be explained by supposing none too careful transcription; but a re-examination of the whole of the Leibnizian remains should include a careful scrutiny on the point as to whether some of the extracts given by Gerhardt are not the work of pupils of Leibniz, whose writing would naturally be somewhat similar. Perhaps too some of those early geometrical theorems might be un-

UMI

earthed; and this would well reward the most painstaking search. Nobody can assert that anything like an adequate tale of the progress of the Leibnizian genius has so far been told.

11 November, 1673.[26]

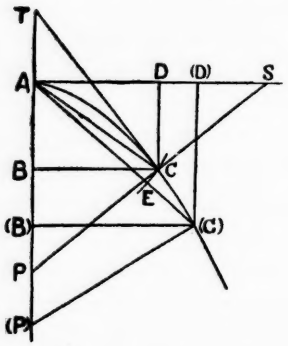
Methodi tangentium inversae exempla.

(Examples of the inverse method of tangents.)

A year or two ago I asked myself the question, what can be considered one of the most difficult things in the whole of geometry, or, in other words, what was there for which the ordinary methods had contributed nothing profitable. To-day I found the answer to it, and I now give the analysis of it.

Find the curve C(C), *in which* BP, *the interval between the ordinate* BC *and* PC *the normal to the curve, taken along the axis* AB(B), *is reciprocally proportional to the ordinate* BC.

Let AD(D) be another straight line perpendicular to the axis AB(B), and let ordinates CD be drawn to it, so that the abscissae



AD along the axis AD(D) are equal to the ordinates BC to the axis AB(B), and the ordinates CD to the axis AD(D) are equal to the abscissae AB along the axis AB(B). Let us call AD=BC=y, and AD=BC=x; also let BP=w and B(B)=z. Then it follows from what I have proved in another place that

[26] See Cantor, III, p. 183; but neither Cantor nor Gerhardt appears to offer any suggestion as to why this date should have been altered.

$$\int wz = \frac{y^2}{2}, \text{ or } wz = \frac{y^2}{2d}$$ [27]

But from the quadrature of a triangle it is evident that $\frac{y^2}{2d} = y$; and therefore $wz = y$.

Now, from the hypothesis, $w = b/y$, for thus w and y will be reciprocally proportional to one another. Hence we have

$$\frac{bz}{y} = y, \text{ and thus } z = \frac{y^2}{b}.$$

But $\int z = x$, hence $x = \int \frac{y^2}{b}$; and from the quadrature of the parabola $\int \frac{y^2}{b} = \frac{y^3}{3ba}$; hence, $x = \frac{y^3}{3ba}$; and this is the required equation expressing the relation between the ordinates y and the abscissae x of the curve C(C), which was to be found. Therefore we consider that the curve has been found and it is analytical; in short, it is the cubical parabola whose vertex is A.

We will therefore see whether the truly remarkable theorem is not true, namely, in the cubical parabola C(C), the intervals BP between the normals to the curve, PC, and the ordinates to the axis, BC, taken along the axis ABP, are reciprocally proportional to the ordinates, BC.

The truth of this is easily shown by the calculus of tangents. For the equation to the cubical parabola is $xc^2 = y^3$; taking c to be the *latus rectum*, and supposing that for c^2 we put $3ba$, or $c = \sqrt{3ba}$, we have $3xba = y^3$.

Now, by Slusius's method of tangents, we have $t = y^3/3ba$, where t is put for BT, the interval along the axis between the tangent and the ordinate.

But $\text{BP} = w = \frac{y^2}{t}$, and therefore $w = \frac{y^2}{\frac{y^3}{ba}} = \frac{ba}{y}$; hence, the w's and the y's are reciprocally proportional as was to be proved.

[27] This was obtained in the form omn.$p = y^2/2$, previous to October, 1674, from the Pascal form of the characteristic triangle; it is quoted as a known theorem in the essay dated 29 October, 1675. See §§ III, VI.

It is probably at this date that he began to revise his ideas as to d diminishing the dimensions; being forced to reconsider them by the occurrence of such equations as $wz = y$. It is seen in the next paragraph how careful he is to keep his dimensions equal; for he introduces an apparently irrelevant $a(=1)$ for this purpose. It gradually dawns on him that neither $\int$ nor d alter the dimensions, but that a "sum of lines" is really a sum of rectangles, on account of the fact that they are applied in a certain fixed way to an axis; he is not quite certain of this however until well on in the next year, when we find him using $\int dx\, y$.

The artifice of this analysis[28] consisted in obtaining the abscissa from the ordinate; and this idea was never previously thought of. It is not a more difficult question either, if the curve is required in which BP, the interval between the normals and the ordinates, is reciprocally proportional to the abscissae AB. Indeed, $w=a^2/x$; but $w=y^2/2$; hence, we have

$$y=\sqrt{2\int w} \text{ or } \sqrt{2\int\frac{a^2}{x}}.$$

Now $\int w$ cannot be found except by the help of the logarithmic curve.[29] Hence, the figure that is required is that in which the ordinates are in the subduplicate ratio of the logarithms of the abscissae; and this curve is one of the transcendental curves.

Now, in truth, it is a much harder question,[30] if the curve, in which AP is reciprocally proportional to the ordinate BC is required.

For then $x+w=\frac{a^2}{y}$ and $wz=\frac{y^2}{2d}$; also $\int z=x$,

or $z=\frac{x}{d}$; thus, $w\frac{x}{d}=\frac{y^2}{2d}$, and $w=\frac{y^2}{2d}\cup\frac{x}{d}$;

hence, $x+\frac{y^2}{2d}\cup\frac{x}{d}=\frac{a^2}{y}$.

If we suppose that the x's are in arithmetical progression then $x/d=z$ will be constant, and we shall have

$$x+\frac{y^2}{2d}=\frac{a^2}{y} \text{ or } \int x=\int\frac{a^2}{y}-\frac{y^2}{2},$$

therefore

$$\frac{x^2}{2}+\frac{y^2}{2}=\int\frac{a^2}{y} \text{ or } d\sqrt{x^2+y^2}=\frac{2a^2}{y};$$

[28] It is difficult to see exactly what Leibniz means by this statement; I can only guess at substitution by means of the theorem $wz=y$, the equivalent to the recognition of the fact that $y\,dy/dx.dx=ydy$. The wording is however impersonal, and may mean that he himself had never thought of the idea before.

[29] Required $y=f(x)$, such that $y\,dy/dx=a^2/x$; the solution is $y^2=2a^2\log_e Ax$. Weissenborn remarks on the omission of the a as being incorrect; from Leibniz's standpoint I cannot agree with him. Leibniz, from Mercator's work, connects a^2/x with the ordinate of the equilateral hyperbola $xy=a^2$, and its integral with the quadrature of this curve. The omission of the a^2 only alters the base of the logarithm, and Leibniz merely states that the solution is of a logarithmic nature without attempting to give it exactly.

[30] How does he know until he has tried it? This rather combats the idea that these were mere exercises; it gives this essay the appearance of being a fair copy intended either for publication or for one of his correspondents. If this were the case, the errors later in algebraical work are all the more unintelligible. The idea that Leibniz was a man who was accustomed to writing down his thoughts as he went along does not appeal to me at all; this is the method of the slow-working mind, rather than that of genius.

but, if we join AC, A(C), then these are equal to $\sqrt{x^2+y^2}$; and if with center A and radius AC we describe an arc CE to cut the straight line AE(C) in E, then E(C) will be the difference between AC and A(C); that is, $E(C) = e = d\sqrt{x^2+y^2}$

$$\therefore e = 2a^2/y.$$

If then it were allowable to assume that the y's were also in arithmetical progression, we should have what was required; yet it seems that it does not make any difference even if the x's have been assumed to be in arithmetical progression. For if we do assume that the x's are in arithmetical progression, it follows that the AD's, or the y's are the reciprocals of the E(C)'s or the e's. Moreover, if they are so at any one time they are so at all times. Also, the sums of an infinite number of reciprocal proportionals, no matter what the progression may be of which they are taken as the reciprocal proportionals; for in this case there is not any consideration of rectangles, where there is need of equal altitudes, but a sum of lines is calculated, that of all the E(C)'s.[31] Hence I see the difficulty arise from the fact that the sum of every e, or every $2a^2/y$, or every E(C), cannot be obtained, unless we know to what progression the y's belong. In this case, that information is not given; for it is necessary that the x's should be in arithmetical progression, and hence that the y's are not so.

On the other hand, if we suppose in the above equation,

$$x + \frac{y^2}{2d} \cup \frac{x}{d} = \frac{a^2}{y},$$

that the y's are in arithmetical progression, then we have

$$x + \frac{y}{dx} = \frac{a^2}{y} \text{ or } xy + \frac{y^2}{dx} = a^2;$$

and, finally, by assigning the progression to neither x nor y, we have in general

$$xy + y\frac{d\frac{y^2}{2}}{dx} = a^2 \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

But we have not as yet really obtained anything. Let us therefore consider it from the standpoint of "indivisibles"; let PCS produced meet AD in S; then the sum of every AP applied to AB

[31] This seems to be the root of the error into which he falls; he has not yet perceived that the e's have to be *applied to some axis*, before he can sum them; and this is to a great extent due to the omission of the dx, taken as constant and equal to unity. He is thus bound to fall back on the algebraical summation of a series.

UMI

is equal to the sum of every AS applied to AD;[32] or calling DS, v, we have

$$dy\int y + dy\int v = dx\int x + dx\int w,$$
$$\text{or } dy\int y + dy\int v = dx\int a^2/y,$$

by the hypothesis of the question.

Now, if we take the y's to be in arithmetical progression, we have

$$\frac{y^2}{2} + \frac{x^2}{2} = dx \text{ Log}y.$$ [33]

But just above, making the same supposition that the y's were in arithmetical progression, we had

$$xy + \frac{y^2}{dx} = a^2 \text{ or } dx = \frac{y^2}{a^2 - xy};$$

and now we have

$$dx = \frac{x^2 + y^2}{2\,\overline{\text{Log}y}}.$$

Hence at length we obtain an equation, in which x and y alone remain, and unshackled, namely

$$\overline{y^2 + x^2},\ a^2 - yx = 2y^2\,\overline{\text{Log}\,y};$$

and this equation, since it is determinate, will give the required locus.

This then is an exceedingly remarkable method, for the reason that when it is not in our power to have as many equations as there are unknowns, yet often we shall be able to obtain some more equations, by the help of which we shall be able to eliminate certain terms, as the term dx in this case, which alone stood in our way. Either of the two equations, by itself, contained the whole nature of the locus, although from neither of them could the solution be derived, because so far easy means were lacking; yet the combination of the two equations gave the solution at once.

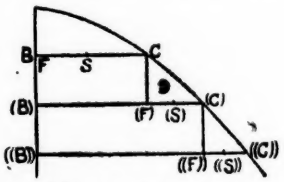
I see that the same thing could be otherwise obtained by moments; and here there comes to my mind a new consideration that is not altogether inelegant.

[32] From the characteristic triangle, AS : AP = dx : dy.

[33] This is of course nonsense. The error seems to arise from the dx being placed outside the integral sign; thus he assumes that dx is constant, while, for the integration, he also assumes that the dy is constant.

We cannot argue from this equation that Leibniz did not at this date appreciate what an infinitesimal was, on account of the infinitesimal being equated to a finite ratio; for since he is assuming that dy is an infinitely small unit, dx really stands for dx/dy.

In the attached figure, let $BC=y$, $FC=dy$; let S be the middle point of FC; then it is evident that the moment of FC is the



rectangle contained by FC and BS, i. e., the rectangle BFC; this follows from the fact that it is equal to BFC+SFC, and the latter can be neglected as being infinitely small compared to the former.[34]

Hence $\int y\,dy=y^2/2$, or the moment of all the differences FC will be equal to the moment of the last term, and $y\,dy=d(y^2/2)$, or $y^2dy=y\,dy^2/2$.

Now, just above, in equation (A), by making x arithmetical, we had

$$y\,d\frac{y^2}{2}=a^2-xy\text{, or }d\frac{y^2}{2}=\frac{a^2-xy}{y};$$

but this is the same thing as $y\,dy$; hence $y\,dy=\frac{a^2-xy}{y}$, and therefore $\int y\,\overline{dy}=\int\frac{a^2}{y}-\frac{x^2}{2}$. But we have already found that $\int y\,\overline{dy}=\frac{y^2}{2}$; therefore $y^2+x^2=2\int\frac{a^2}{y}$, as before; i. e., $d\overline{x^2+y^2}=\frac{2a^2}{y}$.

From this there follows something to be noted about these equations, in which occur $\int$ and d, where one quantity, in this case for instance the x, is taken to proceed arithmetically, namely, that we cannot make a change, nor say that the value of x is found, thus, $x=2(a^2/y)-d\,\overline{y^2}$; for $d\,\overline{y^2}$ cannot be understood unless the nature of the progression of the y's is determinate. But the progression of the y's, in order that it may be used for $d\,\overline{y^2}$, must be such that the x's are in arithmetical progression; hence the dy's depend on the x's, and therefore the x's cannot be found from the dy's. For the rest, by this artifice many excellent theorems with regard to curves that are otherwise intractable will be capable of being investigated, namely, by combining several equations of the same kind.

In order that we may be better trained for really very difficult

[34] Note the advance in ideas suggested by the words "infinitely small *compared with* the former." Here, of course, the notation BFC is the usual notation of the period for BF.FC, the rectangle contained by BF and FC.

considerations of this kind, it will be a good thing to attempt just one more, as for instance when the AP's are reciprocally proportional to the AB's.

Here $x + w = \frac{a^2}{x}$, and $zw = d\frac{y^2}{2}$, and $z = dx$; and so we obtain

$$w = \frac{d\frac{y^2}{2}}{z} = \frac{d\frac{y^2}{2}}{dx}, \text{ hence } x + \frac{d\frac{y^2}{2}}{dx} = \frac{a^2}{x}.$$

The solution of this is not now difficult; for if we suppose that the x's are arithmetical,[35] we have

$$\int x + \frac{y^2}{2} = \int \frac{a^2}{x}, \text{ or } x^2 + y^2 = \overline{\text{Log}y}. \qquad (36)$$

Hence, $\sqrt{x^2+y^2} = \text{AC} = \sqrt{2\,\text{Log}\,\text{AD}}$; and this is a simple enough expression for the curve. In this however the AP's are required to be in arithmetical progression; but on the other hand, if the y's are taken to be in arithmetical progression, we have $x + y/dx = a^2/x$; and from this latter the nature of the curve is not easily obtained.

Let us see whether there can be a curve in which AC is always equal to BP; in this case $\sqrt{x^2+y^2} = w$, and $w = dy^2/2dx$. Let the x's be in arithmetical progression then $(\int\sqrt{x^2+y^2} =)\ \int \text{AC} = y^2$; this, however, is not sufficient to describe the curve practically, that is to say, by points following one another consecutively. When $x = 1$, let $\text{BC} = (y)$; then $\sqrt{1+(y^2)} = (y^2)$, or $1 + (y^2) = (y^4)$. Whence (y) may be obtained; thus, from the equation

$$y^4 - y^2 + \tfrac{1}{4} = 1 + \tfrac{1}{4}, \text{ we have } (y^2) = \frac{\sqrt{5}}{2}, \text{ or } (y) = \frac{\sqrt[4]{5}}{\sqrt{2}}. \qquad (37)$$

Further, in the same way,

$$\sqrt{4 + ((y^2))} + \sqrt{1 + \frac{\sqrt{5}}{2}} = ((y^2));$$
$$\text{AC} \qquad\qquad \text{A(C)}$$

and thus again $((y))$ can be found. By the help of this a third

[35] Note in general that this is Leibniz's equivalent of the modern phrase, "integrate with respect to x."

[36] This I think is more likely to be a slip on the part of Leibniz, than a misprint; for in the next line he has AD, which is the correct equivalent of y. Further, AP varies inversely as x, hence the AP's have to be in harmonical progression, not arithmetical, otherwise x is not equal to $x^2/2$. If on the other hand, we assume three errors of transcription, and replace x for y, AB for AD, AB for AP, the whole thing is correct with an arbitrary base.

[37] It is hardly necessary to point out the error in the arithmetical solution of the quadratic; nor is it important. It is however to be noted that if $\text{AC} = v$, the equation reduces to $v^2 = x(x+v)$, and the solution is a pair of straight lines.

AC can be found, and some sort of polygon can be found, which is more and more like the curve that is required, in proportion as the thing taken for unity is less and less.

That the x's are in arithmetical progression signifies that the motion (in describing it) along the axis AB is uniform. But descriptions that suppose any motion to be uniform are not within our power.[38] For we cannot produce any uniform motion, except a continually interrupted one.

Let us now examine whether $dx\,dy$ is the same thing as $d\overline{xy}$, and whether dx/dy is the same thing as $d\frac{x}{y}$; it may be seen that if $y=z^2+bz$, and $x=cz+d$; then

$$dy=z^2+2\beta z+\beta^2,\ +bz+b\beta,\ -z^2-bz,$$

and this becomes $dy=\overline{2z+b}\beta$.

In the same way $dx=+c\beta$, and hence

$$dx\,dy=\overline{2z+b}\,c\beta^2.$$

But you get the same thing if you work out $d\overline{xy}$ in a straightforward manner. For in each of the several factors there is a separate destruction, the one not influencing the other; and it is the same thing in the case of divisors.

Now let us see if there is any distinction when we seek the sums of these things. We have $\int dx=x$, $\int dy=y$, and $\int d\overline{xy}=xy$. If then we have an equation, $dx\,dy=x$ say, then $\int dx\,dy=\int x$. But $\int x=x^2/2$, hence $xy=x^2/2$, or $x/2=y$; and this satisfies the equation $dx\,dy=x$; for substituting for y its value, $ax\frac{dx}{2}=x$, or $a\frac{x^2}{2}=x$, (39) which is known to be true.

In sums these results do not hold good; for $\int x\int y$ is not the same thing as $\int xy$; the reason is that a difference is a single quantity, while a sum is the aggregation of many quantities. The sum of the differences is the latest term obtained. However, from the sums of the factors we can find the sums of products, not indeed as yet analytically, but by a certain method of reasoning; such as Wallis has done in this class of thing, not by proving them, but by a happy method of induction. Nevertheless to find proofs for them would be a matter of great importance.

[38] This is strongly reminiscent of Barrow, Lect. I (near the beginning) and Lect. III (near the end).

[39] Leibniz, as a logician, should have known better than to trust a single example as a verification of an affirmative rule.
With regard to infinitesimals note the equation $dx\,dy=x$!

Suppose $\int \overline{zy}$ to be the sum that is required. Let $\int \overline{zy} = w$, then $zy = \overline{dw}$, and $y = \frac{\overline{dw}}{z}$, and $\int y = \int \frac{\overline{dw}}{z}$. Similarly, $\int z = \int \frac{\overline{dw}}{y}$. Suppose that $\int y$ is known, $= v$, and that $\int z$ is known, $= \psi$; then $y = dv = \frac{dw}{z}$, and $z = d\psi = \frac{dw}{y}$, and $\frac{dv}{d\psi} = \frac{z}{y}$. From this it would seem to follow that $d\frac{v}{\psi} = \frac{z}{y}$, and therefore that $\frac{v}{\psi} = \int \frac{z}{y}$. Therefore $\int \frac{z}{y} = \frac{\int z}{\int y}$, which is obviously incorrect. [40] Hence it follows that $\int \frac{dv}{d\psi}$ cannot be equal to $\frac{v}{\psi}$.

What then can it be? We have to sum the difference for v divided by the difference for ψ. That is, not every one of the differences for, or the whole of, v is to be divided by each single difference for the ψ; this is not so, I say, because each single one of the first set is only divided by the single one of the other set that corresponds to it, and not by all of them. Therefore $\int \frac{dv}{d\psi}$ is not the same as $\frac{\int dv}{\int d\psi}$, or $\frac{v}{\psi}$. Will not then $d\frac{v}{\psi}$ be something different from $\frac{dv}{d\psi}$? If it is the same, then also $\int d\frac{v}{\psi} = \int \frac{dv}{d\psi}$, that is $\frac{v}{\psi} = \int \frac{dv}{d\psi} = \frac{\int dv}{\int d\psi}$, which is absurd.

Similarly, if we can suppose that $d\overline{v\psi} = dv\, d\psi$, then $\int \overline{dv\psi}$, or $v\psi = \int \overline{dv\, d\psi}$. Now $v\psi = \int dv \int d\psi$; hence, $\int \overline{dv\, d\psi} = \int dv \int d\psi$; which is absurd.

Hence it appears that it is incorrect to say that $dv\, d\psi$ is the same thing as $dv\psi$, or that $\frac{dv}{d\psi} = d\frac{v}{\psi}$; although just above I stated that this was the case, and it appeared to be proved. This is a difficult point. But now I see how this is to be settled.

If we have v and ψ, and they form some quantity, say $\phi = v\psi$ or v/ψ, and if the values of v and ψ are expressed as rationals in terms of some one thing, for instance, in terms of the abscissa x, then the calculus will always show that the same difference is produced, and that $d\phi$ is the same as $dv\, d\psi$ or $dv/d\psi$. But now I see

[40] If Leibniz can see that this equality is "obviously incorrect," what is the use of the argument that has preceded this sentence; for the final result must also be obviously incorrect.

the former can never happen, nor can it come to the latter by separation of parts; for example,

$$x+\beta, \frown x+\beta,, -, x, x, \text{ becomes } 2\beta x,$$

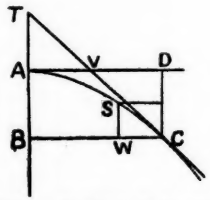
which is quite a different thing from

$$x+\beta, -x,, \frown x+\beta, -x \text{ which gives } \beta^2.$$

Hence it must be concluded that $dv\psi$ is not the same as $dv\, d\psi$, and

$$d\frac{v}{\psi} \text{ is not the same as } \frac{dv}{d\psi}. \quad (41)$$

Take an equation of the first degree, $a+bx+cy=0$. Let $DV=\theta$, $AB=x$, $BC=y$, and $TB=t$. Then, by making use of the method of tangents,[42] we have $bt=-cy$, or $t=-cy/b$. In the same way, $\theta=-bx/c$.



Let $WC=w$, and $WS=\beta$, then it is evident that $t/y=\beta/w$, and

$$\text{therefore } w=-\beta\frac{b}{c}, \text{ and in the same way, } \beta=\frac{-wc}{b}.$$

Second degree. $a+bx+cy+dx^2+ey^2+fyx=0$. Making use of the method of tangents, we have

$$bt+2dxt+fyt=-cy-2ey^2-fyx;$$

[41] Leibniz here justifiably verifies the falsity of his supposition being a general rule by a single breach of it. He uses $v=\psi=x$, and changes x into $x+\beta$; thus,

$$\begin{aligned} d(xx) &= (x+\beta)(x+\beta)-xx &= 2\beta x \\ dx\ dx &= (x+\beta-x)(x+\beta-x) &= \beta^2. \end{aligned}$$

Here we see the first idea of the method that is the same as that used by Fermat and, afterward by Newton and Barrow; this consideration, whatever the source, is that which leads him later to the substitution $x+dx$, $y+dy$ in those cases in which Barrow uses a and e.

[42] "ordinando et accommodando," literally setting in order and adapting. It is to be remembered that Sluse gave only a rule, and not a demonstration of the rule. Part of the rule was that, if the equation in two variables contained terms containing both the variables, these terms had to be set down *on each side of the equation*. Thus, for the equation $y^3=bvv-yvv$ would first of all be written

$$y^3+yvv=bvv-yvv\ldots\ldots\ldots \text{ ordinando (?)}$$

then each term on the left is multiplied by the exponent of y, and each term on the right by that of v, thus,

$$3y^3+yvv=2bvv-2yvv\ldots\ldots\ldots \text{ accommodando (?)}$$

and finally one y on the left, in each term, is changed into a t, where t is the subtangent measured along the y axis.

hence $t=\frac{-cy-2ey^2-fyx}{b+2dx+fy}$. From this it is quite evident that t can always be divided by y (and θ by x), and since $w=\beta y/t$, therefore we have

$$w=\frac{\beta\overline{b+2dx+fy}}{-c-2ey-fx}, \text{ and } y=\frac{-w\overline{c+fx},\frown\beta\overline{b+2\,dx}}{f+2e},$$

but from just above $y=\frac{-a-bx-dx^2}{c+ey+fx}$, hence we have

$$y=\frac{-w,\overline{c+fx},\frown\beta\overline{b+2\,dx},\frown\overline{c+fx},,,+\overline{f+2e}\frown\overline{a+bx+dx^2}}{-w,\overline{c+fx},\frown-\beta\overline{b+2dx},\frown-e} \quad (43)$$

$$=\frac{-w\,\overline{c+fx},-\beta\overline{b+2dx}}{f+2e}.$$

Hence we have an equation in which there is no longer any y;[44] and all figures that can be formed from this equation by a variation of the letters that stand for the constants can be squared; and also all others that by other methods can be shown to be connected with it.

§ IX.

In the manuscript that follows we must refrain from being critical; for, as suggested by the opening remark, it contains nothing more than random notes, jotted down as they came into Leibniz's mind, as materials for further investigation. In the ten days that have intervened since the date of the last MS., he has either had no spare time for further work on the lines of this last manuscript, or else he has found that he cannot proceed any further use-

[43] This is hopelessly inaccurate; all except one error, namely, $f+2e$, which should be $\beta f+2ew$, may be put down to bad transcription. Even if Leibniz's writing were execrable, the correct version of an ambiguous sign (through bad writing) could easily have been settled, *by working through the algebra.* Thus the first of the last pair of values, in Leibnizian symbols should be

$$y=\frac{-w,\overline{c+fx},-\beta,\overline{b+2dx},,\frown\overline{c+fx},,,+\overline{\beta f+2ew},\overline{a+bx+dx^2}}{-w,\ c+fx,\ -\beta,\ b+2dx,,\smile-e,}$$

with a similar correction in the second value.

[44] Even if Leibniz had worked out the correct result, and obtained what he was trying for, namely, w/β in terms of x, he would have got a very lengthy quadratic, and the roots would be quite beyond his power to use at any time. But he convinces himself that he can thus find the quadrature of any conic, or figures that can be reduces to them.

fully until he has perfected the method he had in hand. He therefore reverts to the method of breaking up the figure into triangles by means of a set of lines meeting in a point, coupled with the ideas of the moment and the center of gravity, in order to try to obtain further general theorems for *analytical use*. In this way, he again comes across the differentiation of a product in the form of an "integration by parts"; but he does not recognize in it the differentiation of a product, for he says that as he has obtained this before he can get nothing new from it. He is still wasting his energies over the idea of obtaining dy/dx as an explicit function of x, for the purposes of *integration* or quadratures. The fact that he can use the method of Slusius *as an unproved rule* seems to have hidden from him the necessity of pushing on his investigations with regard to the laws of *differentiation*, or the direct tangent method.

21 November 1675.

Pro methodo tangentium inversa et aliis tetragonisticis specimina et inventa. Trigonometria indivisibilium. Aequationes inadaequatae. ordinatae convergentes. Usus singularis Centri gravitatis.

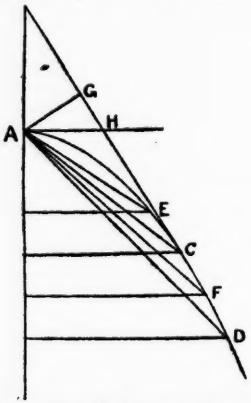
[Examples and discoveries by means of the inverse method of tangents and other quadratures. Trigonometry of indivisibles. Inadequate equations. Converging ordinates. Special use of the Center of Gravity.]

Subject-matter for a new consideration of the Center of Gravity method, as follows:

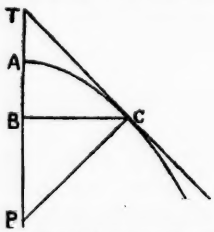
A segment AECD having been broken up into infinite triangles, AEC, ACF, etc., let the center of gravity of each of these triangles be found; this is a simple matter, for the center of gravity is always distant from the base a third of the altitude. Then, since the path of the center of gravity multiplied by the area of the triangle is equal to the solid formed by its rotation, and also since the products of the AH's and the infinitesimal parts of the axis are twice the areas of the triangle, also it is plain that the AG's multi-

UMI

plied by the distances of the centers of gravity of the triangles AEC from the axis are equal to the moment of the segment about the axis; by the help of this idea a number of things can be at once obtained in two ways: first, by taking some general figure and making a general calculation, and then so expressing it that the center of gravity can be easily found; in this way we may obtain the moments of spaces which would be a matter of difficulty otherwise, if they were investigated by the ordinary method of ordinates.



Secondly, on the other hand, if figures of which the moments are easily obtained in the ordinary way are treated by this method, we shall arrive at certain very difficult curves, the dimensions of which can always be deduced from some that are easier. Here then we have a remarkable rule, by the help of which useful properties can always



be obtained from any method however complicated. It is often useful when problems arise that we know are naturally simple, and from other reasons are soluble; for thus many notable cases are discovered. See what Tschirnhaus noted about the Hastarian line.

In irregular problems, such as cannot be treated in a straight-

forward manner or reduced to an equation that is sufficiently determinate, because, say, something has to be done inversely, it is useful to compare several ways with one another, of which the results should be identical. This seems to be useful for the inverse tangent method. Here is a case in point.

The figure, in which BP and AT are reciprocally proportional, is required.

Let $TB = t$, then $AT = t - x$, and $BP = a^2/(t-x)$. If this is multiplied by t, we have

$$\square TBP = ta^2/(t-x) = a^2 + a^2x/(t-x) = y^2$$

hence. $$ta^2 = ty^2 - xy^2,$$

or $t = xy^2/(a^2-y^2)$;[45] and therefore $t/x = y^2/(a^2-y^2)$, or all the t's together equal the moment about the vertex of every $y^2/(a^2-y^2)$.

But from other reasons, all the TP's applied to the axis are equal to the TC's applied to the curve.

Now $t/y = \beta/w$, and therefore $w = \dfrac{\beta y}{t = \dfrac{y^2x}{a^2-y^2}} = \dfrac{\beta\overline{a^2-y^2}}{xy}$.

But $\int w = y$, therefore

$$\int \frac{\beta\overline{a^2-y^2}}{xy} = y \quad \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Further, $$wx = \frac{\beta\overline{a^2-y^2}}{y}, \text{ and } \int \overline{wx} = yx - \int y\beta,$$

hence, $$\int \frac{\beta\overline{a^2-y^2}}{y} = yx - \int y\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots (B)$$

Also $$w = dy,\ dy = \frac{\beta\overline{a^2-y^2}}{xy}, \text{ and therefore}$$

$$xy = \frac{\beta\overline{a^2-y^2}}{dy = w} = \int \overline{y\beta} + \int \frac{\beta\overline{a^2-y^2}}{y}.$$

Now if we suppose that the y's are in arithmetical progression, then $w = dy$ is constant and β is variable;

$$\text{hence,} \qquad \beta = \frac{\int \overline{y\beta + \beta\frac{a^2-y^2}{y}}}{a^2-y^2},\ \overline{d\,\beta\overline{a^2-y^2}} = \frac{a^2\beta}{y}.$$

But from equation (B), $\beta\dfrac{a^2-y^2}{y} + \beta y = \overline{dyx}$

hence, $$\beta\frac{a^2}{y} = \overline{dyx}.$$

[45] There is a mistake in sign; $a^2 - y^2$ should be $y^2 - a^2$; hence the work that follows is also wrong.

We have thus obtained two equations that are mutually independent, the first $\frac{dx}{dy} = \frac{yx}{a+y,\ a-y}$ [46] ..... (1)

and the second $\overline{dyx} = \frac{dx\ a^2}{y}$. ..... (2)

Let us seek to obtain others in addition, such as

$$\int t\,dy = \int y\,dx.$$

Now this furnishes us with nothing new; but $\int tw + \int xw = xy$ or $t\,dy + x\,dy = \overline{dxy}$, and $t = \frac{dx}{dy}\,y$; hence the latter $= \frac{\overline{dxy} - x\,dy}{dy}$. Therefore $\overline{dx}\,y = \overline{dxy} - x\,\overline{dy}$.

Now this is a really noteworthy theorem and a general one for all curves. But nothing new can be deduced from it, because we had already obtained it.

However, from another principle we shall obtain a new theorem; for it is known that the sum of every $BP = BC^2/2$; that is to say, $BP = \frac{a^2}{t-x}$, $t = \frac{\beta y}{w} = \frac{\overline{dx}}{dy}\,y$, and therefore

$$BP = \frac{a^2 dy}{\overline{dx}\,y - \overline{dy}\,x} = \frac{\overline{dy^2}}{2} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

We therefore have two equations, in which dx occurs, namely, the first and the third; by the help of these, by eliminating dx, we shall have an equation in which only one of the unknowns remains shackled; thus from equation (1), we have $dx = \frac{\overline{dy}\ yx}{a^2 - y^2}$, and now from equation (3), we get $\overline{dx}\,y\,\overline{dy^2} - dy\,\overline{dy^2}x = 2a^2\,dy$. Hence,

$$dx = \frac{2a^2\overline{dy} + dy\ \overline{dy^2}x}{y\ \overline{dy^2}}.$$

We have therefore an equation between the two values of dx, in which only the y remains shackled. From this, by assuming

[46] Although the variables are separable, Leibniz does not recognize the fact that he can make use of this. For later he states that the solution of a problem cannot be obtained from a single equation. In this case we have

$$\frac{dx}{x} = \frac{y\ \ dy}{y^2 - a^2} = \frac{dv}{v}, \text{ if } y^2 - a^2 = \pm v^2.$$

Supposing this substitution to have been effected, Leibniz would have concluded that $x = v$, and would have stated that he had solved the problem.

But here again he has made an unfortunate choice, for the origin (A) cannot fall on any of the curves $Cx = v$ or $Cx^2 \pm y^2 = \pm a^2$, which is the general solution of the equation. Hence the problem is impossible.

the y's to be in arithmetical progression, that is that $dy = \beta$ a constant, and $\overline{dy}^2 = z$, and $z = z^2/2 = y^2$; $z = \sqrt{2}\, y = \overline{dy}^2$.[47] Thus we have obtained what was required.

We have here a most elegant example of the way in which problems on the inverse method of tangents are solved, or rather are reduced to quadratures. That is to say that the result is obtained by combining, if possible, several different equations, so as to leave one only of the unknowns in the tetragonistic shackle. This can be done by summing ordinates in various ways, or on the other hand, instead of ordinates, converging or other lines.

Note. If, instead of x or y, some other straight line can be found, either one that is oblique, or one of a number converging to the same point, by the employment of which one only of the unknowns is left in bonds, it may be employed with safety. Take for instance the case of finding the relation for the AP's; here the sum of AP's applied to the axis is half the square on AC. Whenever the formula for the one unknown that is left in shackles is such that the unknown is not contained in an irrational form or as a denominator,[48] the problems can always be solved completely; for it may be reduced to a quadrature, which we are able to work out; the same thing happens in the case of simple irrationals or denominators. But in complex cases, it may happen that we obtain a quadrature that we are unable to do. Yet, whatever it may come to, when we have reduced the problem to a quadrature, it is always possible to describe the curve by a geometrical motion; and this is perfectly within our power, and does not depend on the curve in question. Further, this method will exhibit the mutual dependence of quadratures upon one another, and will smooth the way to the method of solving quadratures. Meanwhile I confess that it may happen that there may be need for a very great number of inadequate equations (for so I call them, when there is need for many to solve the problem, although each alone would suffice provided it could be worked out by itself), in order to completely free one of the unknowns from its shackles. For, unfortunately, a solution cannot be obtained from a single equation, unless one of the terms is free from shackles; and if this term appears oftener, then not unless it is freed at least once. Thus there may be a great number

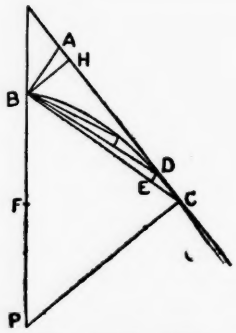
[47] This is quite unintelligible to me as it stands; query, is it an accurate transcription?

[48] This is tantamount to a confession by Leibniz that he cannot explicitly integrate $\int a^2/y$, although he knows that it is logarithmic or reduces to the area under the hyperbola; for he has given this in the MS. for Nov. 11.

of inadequate equations to be found; and we have to examine which of them are in some way independent of the others, i. e., such as cannot be derived from one another by a simple manipulation; for instance, the sum of all the AP's and the sum of all the AE's.

A new kind of Trigonometry of indivisibles, by the help of ordinates that are not parallel but converge.

Let B be a fixed point; let BDC be a very narrow triangle standing upon a curve; let DE be the perpendicular to BC; from the point B let BA, perpendicular to BC or parallel to DE, be drawn to meet the tangent AHDC, and let BH be the perpendicular to the tangent DC produced.



Then the triangles CED, CHB, BHA are similar; hence we have BH/CE = HA/DE = BA/CD, and therefore BH, DE = CE, HA, and BH, CD = CE, BH. Hence it follows that the sum of the triangles or the area of the figure is equal to the products of the AB's into the CE's, or the differences of the ED's and lastly AH, CD = DE, BH.[49]

Further, CH/CE = HB/DE = CB/CD; hence, again, CH, DE = CE, HB, and HB, CD = DE, CB; i. e., the area of the triangle, as is in itself evident, is equal to itself. Lastly, CH, CD = CE, CB; and this last result seems to be worth noting for the case of a Trochoid.

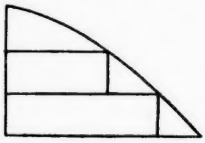
For, if by the rolling of a curve DC on a fixed plane CA, a trochoid curve is described by the point B fixed in DC, and it is given that the ordinate of the trochoid drawn to the fixed plane CA

[49] There are several errors in the letters in this paragraph, which are probably due to transcription; thus, an E for a (? badly written) B, an H for an A, etc., would be quite an easily-imagined error, provided the work was not verified during transcription.

UMI

UMI

is BH, then the sum of the intercepts CH applied to DC will be equal to the sum of the CB's applied to their own differences. Now if any ordinates are applied to their own differences, the same thing is always produced as in the case where we try to find the moment of the differences about the axis, which is the same as the moment when we take the sum of each, or the maximum ordinate, into the



distance of its center of gravity from the axis, i. e., its middle point, that is to say into half itself. Finally this is equal to half the square on the maximum ordinate. Therefore we can always obtain the sum of all the rectangles BC, CE, which is always equal to half the square on BC, or to the sum of all the BP's applied to the axis in F, where CP is the normal to the curve DC.

§ X.

Leibniz now directs his attention to the direct method of tangents, and proceeds to generalize the methods of Descartes. Is it only a coincidence that Barrow uses this method regularly, the curve that he is especially partial to being the rectangular hyperbola? Weissenborn suggests the same coincidence occurs with respect to the method of Newton, who uses analytical approximations; but if there is anything in either of these suggestions. I think that the Barrovian idea, which is purely for the construction of tangents, is much nearer to that of Leibniz in this manuscript than is the Newtonian.

However this may be, Leibniz is at last beginning to consider the point as to the method by which the principle of Sluse is obtained. He ascribes it to a development of the method of Descartes; but in this connection I cannot get out of my head the suggestion raised by Barrow's use of the first person *plural,* "frequently used by *us,*" in the

midst of a passage that is written, contrary to his usual custom, in the first person *singular* throughout, where he describes the differential triangle and the "*a* and *e*" method. I consider that Sluse has enunciated a *working rule* for tangents, which he has generalized by observation of the results obtained by the use of the "*a* and *e*" method; and that this method had been circulated by Barrow some time before the publication of the *Lectiones Geometricae,* although I confess that I have not found any record of this, nor any distinct evidence of a correspondence between Barrow and Sluse; but there is more than a suggestion of this in the fact that Sluse's article was published in the *Phil. Trans.* for 1672.

It seems more than strange to me that there should be such a prolific crop of differential calculus methods within a couple of years of the work of Barrow in all sorts of places, raised by many different people, and that none of them allude to the general seed-merchant, as I consider Barrow to have been.

22 Nov. 1675.

Methodi tangentium directae compendium calculi, dum jam inventis aliarum curvarum tangentibus utimur. Quaedam et de inversa methodo.

[Compendium of the calculus of the direct method of tangents, together with its use for finding tangents to other curves. Also some observations on the inverse method.]

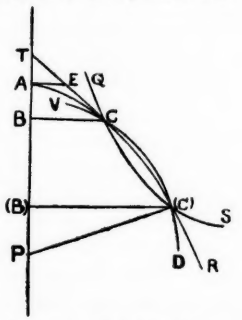
In that which I wrote on Nov. 21, I noted down those things which came to my mind concerning the method of tangents. Returning to the subject, let ACCR and QCCS be two curves that cut one another in one, two, or more points C, C; let AB(B) be the axis; let AB = x be the ordinates, and BC = y the abscissae; then we shall have two equations to the two lines, each in terms of these two principal unknowns. Now if these two equations have equal roots, or the equations have equal values, then the lines will touch one another. Instead of the line QCCS, Descartes chooses the arc of a circle VCCD, whose center is P, so that PC is the least of all

the lines that can be drawn from the point P. It will come to the same thing, and often more simply, if we take not the arc of a circle but the tangent line TC(C), that is the greatest of all those that can be drawn from a given point T to the curve. Let TA = b, AE = e, be assumed as given; required to find AB, BC. The two equations are, the one for the curve AC(C), namely, $ax^2 + cy^2 + \text{etc.} = 0$, and the other to the straight line TC(C) which, on account of the relation TA/AE = TB/BC, will be $b/e = (b \pm x)/y$ or $\pm x = (b/e)y - b$

$$\text{or } y = \pm (e/b)x + e.$$

Thus the value of either one or other of the unknowns can always be obtained explicitly, and thus can be worked out immediately without raising the degree of the equation of the given curve AC(C); and then at once we shall obtain an equation giving the unknown that alone remains, so that we may determine the condition for equal roots. Doubtless this is the principle of Sluse's method.

If however the arc of the circle whose center is P is used, following Descartes, then the new equation, for the circle, will be as follows: let the radius PC = s, and PB = $v - x$, and we have



$s^2 = y^2 + v^2 + x^2 - 2vx$. Hence it is clear that we have the choice of either a circle or a straight line; and when, in the equation to the given curve, only an even power of y appears (as can always be made to happen in the case of the conics), then it will be more convenient to use equations to circles; for thus, by the help of the two values of y^2, the unknown x can be immediately worked out; but, in general for all equations to curves expressed by a rational relation, the method of the straight line may be usefully employed.

Hence I go on to say that not only can a straight line or a circle, but any curve you please, chosen at random, be taken, so long as the method for drawing tangents to the assumed curve is

known; for thus, by the help of it, the equations for the tangents to the given curve can be found. The employment of this method will yield elegant geometrical results that are remarkable for the manner in which long calculation is either avoided or shortened, and also the demonstrations and constructions. For in this way we proceed from easy curves to more difficult cases, and an equation to a curve being supposed known, it is always possible to choose an equation to some other curve whose tangents are known, by the help of which one of the unknowns can be worked out very easily.

Thus, if it is given that $hy^2+y^3=cx^3+dx^2+ex+f$ is the equation to a curve of which the tangents are required, assume a curve of which the equation is $hy^2+y^3=gx+q$, for that of which the tangents are already known; eliminating y, we have an equation such as $gx+q=cx^3+dx^2+ex+f$. This can be determined for two equal roots, either by Descartes's method of comparisons, or Hudde's by means of an arithmetical progression; and thus by working out the value of x, the value of either g or q may be found; and one of the two letters q or g can be chosen arbitrarily.[50] Hence, a way of describing that other curve that touches the given curve is obtained; now, when this is described, let the tangent be drawn at the point which is common to it and the proposed curve, which tangents we have supposed to be already known; then this tangent will touch the given curve.

I think that, in general, the calculation will be possible by this method of assuming a second curve, as we have done in this case, which evidently works out one of the unknowns. Hence I fully believe that we shall derive an elegant calculus for a new rule of tangents, which in addition may be better than that of Sluse, in that it evidently works out immediately one of the two unknowns, a thing that the method of Sluse did not do. Now this very general and extensive power of assuming any curve at will makes it possible, I am almost sure, to reduce any problem to the inverse method of tangents or to quadratures. Indeed let any property of the tangents to a curve be given, and let the relation between the ordinates

[50] The method of Hudde appears to be similar in principle to that of Sluse, while that of Descartes was the construction of the derived function by assuming roots, forming the sum of the quotients of the function divided by each of the assumed root-factors in turn, and comparison with the original function. Both therefore reduce to finding the common measure of the equation to the curve (where the right-hand side is zero) and the differential of it.

Leibniz, however, strange to say, does not note that by taking one of his arbitrary constants, q, equal to f, the equation has its degree lowered in the particular case he has chosen.

and the abscissae be required. Then an equation can be derived, which will contain the principal unknowns, x, y, and always two others as incidentals, such as s and v, or b and e, or the like; now, as the equation contains the property of the tangents, by which s and b may be expressed so as to have a relation to the tangents, assume in this case any new curve chosen arbitrarily, and then s and v will also have some known relation to this curve. By means of the equation to the arbitrarily chosen curve, we shall be able to replace the given property of tangents in favor of the curve required, namely, by removing one or other of the unknowns; and by thus reducing the problem to such a state the inverse calculation will come out the more easily.

The whole thing, then, comes to this; that, being given the property of the tangents of any figure, we examine the relations which these tangents have to some other figure that is assumed as given, and thus the ordinates or the tangents to it are known. The method will also serve for quadratures of figures, deducing them one from another; but there is need of an example to make things of this sort more evident; for indeed it is a matter of most subtle intricacy.

The manuscripts mentioned above seem to be all that were found by Gerhardt belonging to the period 1673-5. I feel that it is a great pity that they were not given in full, or at least a little more fully. For instance, Gerhardt mentions that Leibniz in the MS. of August 1673 constructs the so-called characteristic triangle, but does not give Leibniz's figure in connection. This figure should have been given; for the figure given in October 1674 is not the characteristic triangle as given by Leibniz in the "postscript" (§ I), or the *Historia* (§II), but it *is the Pascal diagram* (assuming that the figure given by Cantor is the correct one). It would be useful to know the date at which Leibniz drops the Pascal diagram in favor of one or other of the Barrow diagrams.

It is to be noticed at this date that Leibniz uses one infinitesimal only, and *verifies* that the method of Descartes comes out correctly in the simple case of the parab-

UMI

ola; but he is not satisfied with the generality of the method of neglecting the vanishing quantities.

Again, the second manuscript of October 1674 appears to be immensely important; especially as it contains the groundwork of some of the later manuscripts. Judging by the little that is given of it, it would seem to be most desirable that fuller extracts, at least, should have been given. It is a matter for remark that this manuscript is a long essay on series. Can this possibly have had anything to do with the fact that it is not given in full?

(TO BE CONTINUED.)

J. M. CHILD.

DERBY, ENGLAND.

CRITICISMS AND DISCUSSIONS.

MECHANISM AND THE PROBLEM OF FREEDOM.

I.

Men's opinions are far more commonly the result of the general presuppositions and prejudices of the age in which they live than the outcome of a rational process. Thus men believe whatever fits in with their general view of life and dismiss without a hearing anything which conflicts with it. In this age of science the scientist has become the arbiter of all questions, and his view is commonly accepted as authoritative. Hence problems which he refuses to examine, e. g., the question of the existence of ghosts, are at once relegated to the realm of superstition. Now there is some danger of freedom being placed among such problems. An indication of this is found in the following words of Haeckel, which represent the attitude of many contemporary scientists and psychologists toward the question of freedom: "The great struggle between the determinist and the indeterminist, between the opponent and the sustainer of the freedom of the will, has ended to-day, after more than two thousand years, completely in favor of the determinist. The human will has no more freedom than that of the higher animals, from which it differs only in degree, not in kind.... We now know that each act of the will is as fatally determined by the organization of the individual and as dependent on the momentary condition of his environment as every other psychic activity."[1] This view has won its way by its scientific prestige, and has been eagerly accepted by many who have never examined the evidence for freedom, but who nevertheless smile indulgently at those who are still so benighted as to believe in it. Therefore, in view of the great popularity and influence that Haeckel has enjoyed it seems profitable to examine the question of the relation of mechanism and freedom with reference to his specific teaching.

[1] *The Riddle of the Universe,* p. 130f.

There are two main principles on which Haeckel bases his system: the doctrine of evolution, and the "law of substance." The doctrine of evolution furnishes the principal evidence for his denial of a spiritual principle in man and together with the "law of substance" leads to his mechanistic determinism. We shall therefore first consider the question whether the existence of a spiritual principle is precluded by evolution or by any other arguments suggested by Haeckel. We shall then proceed to the question of the universality of the "law of substance" and to the problem of its relation to mechanism, and finally examine briefly the adequacy of mechanism itself as a philosophic explanation of the universe.

II.

Haeckel never tires of citing facts in support of the doctrine of evolution. Since this doctrine in some form or other is now almost universally accepted as true we need not stop for an instant to inquire concerning the adequacy of this evidence. The only question for us to consider is whether the doctrine of evolution inevitably leads to the reduction of mind to matter. Put very simply Haeckel's argument for this conclusion is that since man developed from the lowest forms of life there is no reason to attribute to him a separate immaterial or spiritual principle not found in the forms of life from which he originated. Many objections to this argument at once suggest themselves to the thoughtful reader. In the first place it takes for granted the old scholastic idea of rigid continuity, according to which nothing new can ever arise. Now there are grave difficulties in this view, but even waiving these for the moment, Haeckel's conclusion by no means follows. Rather, the doctrine of continuity, if strictly held, would force him to read into the life of the lowest organism all the complex processes and meanings which have been evolved in the highest forms of life. For if evolution were rigidly continuous the very fact that certain phenomena, such as sensation and will, have developed in the later stages of the evolutionary process would show that these phenomena were implicit in the earlier forms. Thus Haeckel would be compelled to understand the protozoon in the light of man, rather than to reduce man to the level of the protozoon. He indeed seems sometimes to do this, and with an extraordinary anthropomorphism bestows elementary will and elementary emotion upon even inanimate matter.[2] If he held consistently to this view his final system

[2] Cf. *infra*, p. 304.

would be in the nature of a theological hylozoism rather than a strictly mechanistic determinism. However, this reading of continuity, this attribution of man's processes to the lower forms of life, is misleading, as it involves what Baldwin calls "the fallacy of the implicit." As a matter of fact, if progress is genuine, new processes and new meanings must arise which cannot be interpreted in terms of the lower stages. Thus even though life has arisen from the inanimate, and consciousness from the unconscious, yet they involve meanings and processes which cannot be expressed in terms of the stages from which they arose. It is fallacious either to deny these new meanings and attempt to reduce them to earlier stages, or to read them back as implicit in the earlier stages. Hence the doctrine of evolution in no wise militates against the spiritual nature of man. Rather it leads us to expect man's nature to be higher or more developed than the merely physical or the merely biological.

In addition to the argument drawn from evolution, Haeckel adduces several other considerations in support of his denial of the spiritual nature of man. He brings forward the evidence of experiments which have shown that various functions of the soul, such as speech and sense images, are connected with definite areas of the cortex of the brain and disappear when these areas are diseased or destroyed.[3] Again he calls our attention to the close connection between man's higher cerebral functions and purely physiological processes—a connection especially plain in the case of emotions.[4] He also emphasizes facts concerning the individual's development which, in his opinion, indicate that the soul originates, grows, and decays with the body.[5] Finally he points out that we never find a single instance of a spiritual principle unconnected with a physical substrate.[6]

Now although all the above arguments show clearly that there is some relation between mind and body, yet they do not succeed in reducing mind to body. The facts can be read as easily the other way. As a matter of fact, we are as conscious of the influence of mind on body as of body on mind. It is true that illness or various physical causes affect man's mental processes, but it is also true that man's mental processes affect his physical condition. In-

[3] Cf. *Last Words on Evolution*, 98f; *The Riddle of the Universe*, p. 204.
[4] Cf. *The Riddle of the Universe*, pp. 127, 204.
[5] Cf. *The Riddle of the Universe*, Chap. VIII.
[6] Cf. *Ibid.*, pp. 90, 91.

UMI

deed every act of will is evidence of the power of mind over matter. Now to this Haeckel might retort: "What you call will is merely a certain functioning of a physiological organism. I can even disclose to you, with my microscope, the minute structures in the brain by which willing takes place." But even if this last contention were granted it would not prove Haeckel's point. The brain with its various structures may be the instrument of mind's expression without being the *cause* of mind.[7] Moreover the very facts of pathology which are cited by Haeckel to show the dependence of mind on matter are used by Bergson to prove that mind cannot be located in the brain nor determined by it.[8] Furthermore in this controversy concerning the relation of mind and body, the idealist can always go back to Berkeley's position and retort, "The brain, the nervous system, etc., to which you attempt to reduce mind, are known only as ideas of mind, and cannot be proved to exist apart from mind."

A further weakness in Haeckel's arguments is that they often betray a total misunderstanding of his opponents' position. They are all directed against the existence of a separate, immaterial substance or soul. Most idealists, however, regard the soul as activity or functioning, rather than as substance. They do not insist on the separateness of the psychic principle or on the existence of any disembodied spirit, but rather on the fact that man's activity cannot be explained in purely physical or physiological terms. Suppose it be granted to Haeckel that the soul is but the "sum total of physiological functions," yet the problem of the activity of the soul is not thereby solved. Consciousness is a fundamental fact of experience, and it cannot be explained by being set aside or labeled an epiphenomenon. Therefore the materialist must explain not only how the body reacts, but how it is conscious, how it thinks, evaluates, loves, struggles, and sacrifices. It is indeed questionable whether this activity can be interpreted in purely biological or physiological terms. The so-called body becomes equivalent to the mind and demands the same sort of an explanation.

We conclude then from this discussion that Haeckel's reduction of the psychical to the physical is not valid, and we turn to an

[7] Cf. Schiller, *Riddles of the Sphinx*, pp. 293ff; Bergson, *Matter and Memory*, pp. 299ff; James, *Human Immortality*, pp. 7-29.

[8] Cf. Bergson, *Matter and Memory*, Chapter II.

examination of the "law of substance"—the second main support of Haeckel's system.

III.

The "law of substance" is a combination of the well-known scientific laws of the "conservation of matter" and "the conservation of energy." According to Haeckel, these laws are but two aspects of one great cosmic law, since they relate to the two inseparable attributes of substance. In passing it may be noted that little is gained by this combination of the two laws, since Haeckel's unknown substance is incapable of showing concretely the relation between matter and energy. Haeckel regards this law as the one great eternal cosmic law. On what evidence then can he base its validity?

The evidence for the law adduced by Haeckel lies in the realm of scientific experiment. Thus the law of "conservation of energy" rests on the fact that many experiments have shown that when one form of energy is changed into another, it may be reconverted into the original form of energy with only a slight loss due to the escape of part of the energy into an unavailable form. Similarly the law of the "conservation of matter" rests on experiments which have demonstrated that the weight of a substance does not change throughout a series of chemical transformations. Moreover no experiments have given any indication of the creation or destruction of matter or of energy, and the generalization of a great number of phenomena under these laws has been indeed a great achievement of science. Yet it is one thing to regard these laws as useful generalizations for the purposes of science, and quite another to erect them into ontological and absolutely universal laws. Against this latter proceeding, which is that of Haeckel, an emphatic protest must be made. There are three grounds for this protest: (1) the laws have never been proved to hold exactly in any field; (2) the fact that the laws appear to hold in one or two fields is no justification for the assertion that they must hold in all fields; (3) experience can never *prove* the absolute universality of any law.

In the first place, it is manifestly impossible to prove that the laws hold exactly in any field, since the inaccuracy of scientific instruments is such that *small* differences might pass unnoted. Furthermore there are always extraneous circumstances which must be taken into account in an appraisal of the results of an experiment. A scientific result is always an approximation, and the scientific

law states what would take place under ideal circumstances rather than what occurs in any concrete situation.

In the second place, the fact that the laws appear to hold true within certain fields of our experience does not show that they *must* hold in all fields. Thus the demonstration of the laws in the case of physical and chemical changes would furnish no proof of their applicability to the relation between the physical and the psychical. It is at this point that Haeckel's assertion of the universality of the law depends upon his reduction of the psychical to the physical. Since this is not valid, he is not entitled without more ado to extend the application of the law to the psychical realm. The application of the law here must rest upon experiments showing that a certain amount of physical energy can be transformed into a definite amount of psychical energy and reconverted into the original amount of physical energy. Manifest difficulties stand in the way of such experimentation, but until something of the sort is carried out there is little significance in speaking of the psychical life as a form of energy. To do so merely covers up the fact of our ignorance concerning the relation between the psychical and the physical. Now apparently Haeckel himself is aware of some of the difficulties in the way of regarding the psychic as a form of energy since in his last work, contrary to many of his previous assertions,[9] he explicitly teaches that the psychic is a separate attribute of substance, coordinate with matter and energy.[10] If this be admitted, however, the psychic must demand its own law of the "conservation of the psychic," if it is to come under the "law of substance." In any case, the important point for our purpose is that until the law is proved to be valid in the psychical field it furnishes no ground for a denial of freedom.

Our last objection needs no justification, as it is a philosophic commonplace that laws resting on experience can be universalized only by means of the supposition of the uniformity of nature. This uniformity, however, cannot be proved by experience without the assumption of its own existence in the attempted proof. Thus the observation that the laws apparently hold in a comparatively few instances within the narrow range of our experience is no proof that they have always held and will always hold throughout the length and breadth of the universe.

We have seen reason to question the dogmatic assertion of the

[9] Cf. *The Riddle of the Universe*, p. 220; *Anthropogenie*, p. 941.

[10] Cf. *Die Lebenswunder*, p. 185.

universality of the "law of substance." Yet if it be admitted for the sake of argument that the law is universal and necessary, it by no means follows that this law alone gives an adequate account of reality and a solution of all its riddles. The law is an abstraction; it is purely quantitative, and as such leaves out of account the qualitative aspects of the universe. Thus although the *amount* of matter and energy in the universe remain constant, changes in their form or in their combination bring about new qualities not reducible to mere quantity. Take, for example, the case of a chemist who mixes together two elements in a new combination. Their weight, as a measure of their quantity, remains the same, but this quantitative equality in no wise explains or describes the new odor, the new color, or other new properties possessed by the compound. These qualitative aspects, however, are certainly part of reality, although they cannot be described by the law of substance nor comprehended in a system which uses this law as the solution of all its problems.

The "law of substance" is for Haeckel but a necessary consequence of mechanical causation. In fact the two for him are identical.[11] Yet the relationship does not appear to be as simple as he would have us believe. The law of substance, he tells us, is a consequence of mechanical causation, yet his proof of the latter rests largely on his supposed proof of the universality of the former. Now the law of mechanical causation, involving the equivalence of past and present, might lead naturally, though perhaps not inevitably, to the "law of substance." On the other hand the "law of substance" does not necessarily involve mechanical causation. It does indeed preclude spontaneity, but it would be as compatible with teleology as with mechanism, since it says nothing concerning the origin of changes in matter or energy. The amount of energy and matter in the universe might remain constant if their changes were due to a desire for a future state as well as if they were due to a past stimulus. Thus even the universality of the "law of substance" would not prove the universality of mechanism. The latter theory must stand on its own feet and be accepted or rejected on its own merits.

IV.

Haeckel declares that mechanical causation explains all phenomena. To quote his own words: "The great abstract law of mechanical causality, of which our cosmological law—the law of substance—is but another and a concrete expression, now rules the

[11] Cf. *The Riddle of the Universe*, pp. 215, 366.

entire universe as it does the mind of man; it is the steady immovable pole-star whose clear light falls on our path through the dark labyrinth of the countless separate phenomena."[12] "The monism of the cosmos which we establish thereon proclaims the absolute dominion of 'the great eternal iron laws' throughout the universe. It thus shatters at the same time the three central dogmas of the dualistic philosophy—the personality of God, the immortality of the soul, and the freedom of the will."[13]

Before accepting Haeckel's conclusion concerning freedom the adequacy of mechanism itself must be examined. Of the many objections which might be made, and which have been made, to universal mechanism, we shall confine ourselves to the following: (1) the universality of mechanism cannot be proved; (2) the universality of mechanical causation would not, as Haeckel would have us believe, necessarily preclude purpose and rational or ethical freedom; (3) mechanism by itself fails to give a satisfactory account of experience as we actually know it.

We contend that mechanism cannot be proved. Experience cannot show that mechanical causation is universal and necessary, and reason does not disclose any logical necessity for insisting that every aspect of reality shall be explained by reference to the past. On the contrary, the concept of mechanical causation is full of difficulties which force the mind beyond it.[14] The universality of mechanical causation is indeed a methodological postulate of science, but not necessarily a universal principle of reality. Haeckel makes many dogmatic assertions to the effect that mechanism is universal, and that even the will is absolutely bound by causal law. Thus the will is, he declares, the necessary outcome of heredity and environment. Yet obviously he cannot *prove* that such is the case. He cannot prove that A did a certain act because A had a certain heredity and a certain environment, and that A could not have done anything else. Indeed, in the case of human activities so many complex conditions occur that it is practically impossible to isolate any set of conditions in such a way as to establish a uniform series of cause and effect. Hence the establishment of causal connection (quite apart from the question of its universality and necessity) is in such cases a task for the future, rather than

[12] *The Riddle of the Universe*, p. 366.

[13] *Ibid.*, p. 381.

[14] For a careful analysis of causation cf. Taylor, *Metaphysics*, pp. 158ff; Ward, *Realm of Ends*, pp. 273ff; Bergson, *Time and Free Will*, pp. 199-221.

an accomplished fact. The possibility therefore remains that the mental life may resist such causal treatment. An indication in this direction is found in the comparative lack of success of psychology in the use of scientific methods found fruitful in other fields.

In the second place, even though mechanism were proved to be universal, this would by no means preclude the possibility of purpose, of value, and of rational or ethical freedom. Haeckel's absolute denial of all distinctions of value is evident in the following quotations: "As our mother earth is a mere speck in the sunbeam in the illimitable universe, so man himself is but a tiny grain of protoplasm in the perishable framework of organic nature."[15] "Our own 'human nature,' which exalted itself into an image of God in its anthropistic illusion, sinks to the level of a placental mammal, which has no more value for the universe at large than the ant, the fly of a summer's day, the microscopic infusorium, or the smallest bacillus. Humanity is but a transitory phase of the evolution of an eternal substance, a particular phenomenal form of matter and energy, the true proportion of which we soon perceive when we set it on the background of infinite space and eternal time."[16] From Haeckel's point of view, indeed, neither man nor the bacillus can have any value for the "universe at large," since there is, in his opinion, no purpose whatever in the universe. All is but the result of blind forces, and even the progress of evolution is of no value to the universe.[17] Now this denial of value is explained by the fact that Haeckel always associates teleology with a separate immaterial principle. He regards it as an interruption of mechanical causation, and so feels that it is incompatible with monism. Whether or not an absolute monism of any sort is compatible with distinctions of value and with freedom, at least it is plain that the denial of teleology does not necessarily follow from the establishment of mechanical causation. Mechanism, as many teleologists tell us, may be the instrument of purpose. Far from being antagonistic to teleology it alone makes teleology possible. Without it purpose would be impotent. For example—to take an analogy from human life—man can utilize natural processes for the carrying out of his purposes only in so far as he can rely upon their mechanical uniformity. Even a machine is an embodiment of purpose. It works in a mechanical way, but its construction can be

[15] *The Riddle of the Universe*, p. 14.
[16] *The Riddle of the Universe*, p. 244.
[17] Cf. *Ibid.*, Chap. XIII.

explained only in terms of purpose. From this point of view, teleology is not an external principle opposed to mechanism, but rather is immanent in all natural processes, and includes and transcends their merely mechanical aspects. The processes of the universe are describable in terms of mechanical causation, but these series of mechanical changes are what they are by reference to their value for the whole.

Again we object to mechanism taken as a sole explanation of the universe on the ground that it fails to take into account many facts of experience. Although supposed to be the direct outcome of an acceptance of evolution, mechanism has been unable to give a satisfactory explanation of evolution itself. Furthermore mechanism cannot explain the existence of values, purposes, and ideals, and many other aspects of reality.

Bergson, perhaps better than any one else, has succeeded in proving the first point. In his careful examination of theories of evolution, he shows how mechanism is forced to take refuge in a miracle to account either for the successive production and preservation of millions of minute variations in the same direction, or for the complementary changes of the various parts of an organ necessary for the preservation and improvement of its functioning. Moreover this same miracle must be repeated innumerable times as the same change has taken place in many different lines of evolution.[18] Furthermore, in every explanation of evolution, terms such as adaptation and struggle for existence occur, but these are not mechanistic terms, since they imply purpose, ends, value. The mechanist holds that all achievements of evolution are merely results of external and internal forces, which are absolutely blind. Yet if such is the case, why is the organism said to struggle for existence? Haeckel himself, indeed, often finds a place for the action of internal forces and declares that the movement of molecules is due to an inner will. "Even the atom is not without a rudimentary form of sensation and will, or, as it is better expressed, of feeling and inclination—that is, a universal 'soul' of the simplest character."[19] The term "inclination" suits Haeckel's purpose by its vagueness, but if it is at all comparable to will it implies a reaching for the future which is not explicable as merely the result of a previous force. To do justice to this "inclination" Haeckel would be forced beyond his rigid determinism.

[18] Cf. Bergson, *Creative Evolution*, pp. 62-76.

[19] *The Riddle of the Universe*, p. 225.

UMI

UMI

The discussion of the inadequacy of mechanism as an account of evolution has led directly to our second criticism: that mechanism fails to do justice to the existence of values and purposes which are present not alone in our inner experience but which find an outer embodiment in the great achievements of civilization.[20] Surely the painting of a great picture, the writing of a drama, or the founding of a college cannot be accounted for as the result of purely natural forces. Now the mechanist, of course, does not attempt to deny the presence and power of ideals in human life. His contention is simply that these ideals themselves are the result of purely mechanical forces. Will, Haeckel says, is absolutely determined.[21] Thus, according to Haeckel, the psychologist can trace the behavior of the self to causes in preceding conditions much as the physicist traces causal connections between the motions of stones. Haeckel, however, overlooks the fact that at this point we happen to be in a peculiarly favored position. We *can* see the action from within as well as from without, and as we do so we discover a process of determination differing profoundly from the mode of determination described by the scientist. In our own case our action cannot be understood apart from our ideal. This ideal, although due to preceding conditions of one sort or another, does not act upon us as an external compelling force, but influences us through the appeal it makes to our own interests. It is an ideal for us because we ourselves select it, and not because it is forced upon us by any external force. But this process of the selection of an ideal, or of evaluation, is distinct from any process found in the purely physical world and is not describable in mechanistic terms.[22]

Haeckel himself grows eloquent over the ideals of the good, the true, and the beautiful, and urges us to put these ideals before any false ideals promulgated by superstition. Such exhortations, however, have apparently little place in an absolutely mechanistic scheme where each self is absolutely determined by his heredity and environment. Haeckel becomes indignant over what he regards as superstitions, yet he should recall that all superstitions as well as the despised dualistic philosophy are, on his scheme, natural products and therefore as necessary as his own monistic utterances.

[20] Cf. Bosanquet, *The Value and Destiny of the Individual*, pp. 109ff.

[21] Cf. *The Riddle of the Universe*, p. 131; *History of Creation*, Vol. I, p. 237.

[22] For a clear description of the distinction between personal and mechanical determination, cf. Ward, *Realm of Ends*, pp. 179ff.

UMI

Furthermore the ideals of the good, the true, and the beautiful must be, for Haeckel, purely human ideals, since no values exist for the universe. But if man himself has as little value as Haeckel gives him, it is strange that he should regard human ideals as worthy of reverence and worship.

A final word must be added to our criticism of mechanism. The theory of mechanism itself is not, as Haeckel must believe, a purely natural product. It is due to the organizing activity of man's intelligence and could not exist without it. Haeckel regards this unifying and critical faculty of man as due to the "concatenation of presentations."[23] Yet the mere concatenation of presentations could never of itself lead to the criticism and combination necessary to bind together these various sensations under the law of causation. This unifying of experience demands, as Eucken has so clearly shown, that man be able to separate himself from the chain of nature in order to combine and order the presentations that come to him. Hence the formulation of the theory of mechanism is a fact which mechanism itself fails to explain, and the very existence of the *theory* is evidence of its own inadequacy as a final explanation of all facts in the universe.

Our examination of Haeckel's philosophy has shown the lack of cogency of his denial of freedom. While this in itself furnishes no evidence for the reality of freedom, it at least frees us from many objections that are commonly raised against it. It indicates that the problem cannot be disposed of in so summary a manner by science, and thus affords ground for those who in the twentieth century, in spite of Haeckel's dictum, maintain the possibility of freedom.

GERTRUDE CARMAN BUSSEY.
GOUCHER COLLEGE, BALTIMORE, MD.

DETERMINISM OF FREE WILL.

WITH REFERENCE TO THE PRECEDING ARTICLE.

There is a strange confusion about mechanicalism and freedom of will which seems to have been constructed by our theological school of educators on the basis of a misinterpretation of philosophical thought, and errors thus derived are still perpetuated.

The idea of the will is perhaps the fundamental conception of

[23] Cf. *The Riddle of the Universe*, pp. 121f.

ethics, and an important item for moral purposes is the freedom of an acting person. But "free will" is nothing mysterious nor incredible; it is that condition of a will which is not hindered by compulsion. He is free who acts on his own account, according to his own character, and is not interfered with by external circumstances which would make it impossible for him to act as he wishes. A man under compulsion is not responsible for his action; for his act is the act of some one else, or is due to the circumstances which force him against his own will. The external circumstances may be ever so indirect and may be reducible to fear. A man threatened by the consequences of the results of his act is no freer than a man who is directly forced into acting contrary to his will by facing the revolver of a highway robber. If an act is committed because the acting person wishes the act and also willingly accepts all of its consequences, it is and ought to be considered an act of free will, and there is scarcely any thinker who would not admit this definition of free will.

Is there any one who denies that the act of a free will, as here defined, is as much determined as any other event in this world in which we live? If the free act of a man is really the result of deliberation and if it is performed according to the nature of the actor's character, the result of this decision will be as necessary as the act of an unfree man who acts under compulsion according to motives of fear or any external force. Determinism is a general feature of the world which expresses the truth that the law of causation remains unbroken. According to the law of causation, everything is determined, even the act of a free man.

Yet there are, or rather have been, some theologians who believe in free will, not as free will necessarily must be, viz., an unhampered will, but as a *carte blanche* or *tabula rasa*, a cause that is not caused, or as a determinant which on its part is undetermined, which is free in the sense that it is unformed, or a factor that is somehow an exception to causation and not the product of the efficacy of causation. They think that a man is not responsible if his actions are determined or determinable and can be predicted, just as in moving pictures only such consequences will happen as are on the films, and the man who knows the film would naturally and necessarily be able to tell what is going to happen in the next moment. What an undetermined will is or would be, has never as yet been clearly described; it is only declared to be an exception to the law

UMI

of causality, and being undetermined seems to be as much a mere chance product as the haphazard cast of dies, in which case of course the actor could no longer be regarded as responsible for a deed not determined by himself.

The truth is that if we were omniscient we could predict the history of the world from step to step just as the theatrical manager of the movies knows the next act if he knows the film that is to project it on the screen. If I know all the characters of the acting persons, I will be able to predict the outcome of their activity under definite conditions, and there can be no quibbling about it.

We must not identify necessity and compulsion. Everything is determined; and all acts are determined with necessity, even the free acts of a free man. Further it would be wrong to say that man is compelled to act according to factors which are none of his making, if he necessarily acts according to his will.

It is true that there are factors which have preceded him; among them there are factors such as have determined his character. He has been determined and his will has been given him. In this sense it is claimed that he is as unfree as any slave who is not his own master. But is that not a wrong conclusion that here too identifies necessity with compulsion? It is necessary that a man should act according to his character if he is not under compulsion. The acts of a free man are necessary because his will necessarily and naturally follows the impulses of his own character. To say that we are slaves because we follow necessarily our own instincts is simply an illogical distortion of facts. The truth is that in doing what we will we obey the behest of those factors which shaped our will. However, granting that our will is not of our own making, we will be obliged to confess that we are the continuation of those factors which make us; or in other words, our ancestors whose will we incorporate are ourselves in a former generation. Thus we ought to recognize openly and unhesitatingly that the whole development of the world is not a piecing together of independent individuals, but that we are mere fragments of a continuous whole, we are pieces of a prolonged history of one and the same aspiration which may be modified, improved, or even on the other hand weakened and debased. Former generations have made us of the present age, and future generations will be as much the product of the present generation as we are of the past. Thus if we speak of having been made by prior factors we must recognize

that the factors that made us are our own existence, as we existed in former days,—yet the truth remains that a free will is definitely determined. A free will which acts in an unhampered way is as much determined as any will which suffers violence or acts under compulsion.

Miss Bussey has taken up Haeckel and criticizes him for denying freedom of will where he stands up for determinism. I do not think that Professor Haeckel will take up the cudgel and defend himself. On the other hand I grant that Professor Haeckel is an enthusiastic defender of the monistic world-conception for which he demands a strict and universal application of the mechanistic theory to all events of existence. I will not deny that Professor Haeckel sometimes accepts views which I myself would not endorse. For instance he identifies God with matter and energy while I would look upon God in contrast to matter and energy, as a religious formulation of the world order which is the ultimate *raison d'être* of natural law throughout the sphere of existence, including also the natural law that governs human society and is the basis of the rules of conduct. But this is a point which could easily be reconstructed or altered, for Professor Haeckel himself would scarcely object to it.

In order to understand Haeckel one ought to interpret his writings in the spirit in which he has written them, and ought not imply mistakes which are rather incidental points, such as Miss Bussey criticizes.

Miss Bussey in criticizing Professor Haeckel should consider that he rejects the theory of free will because he understands by free will the theological conception of an undetermined will, viz., that kind of a free will that does not exist, because it is a self-contradictory notion, an impossible and foolish conception of a misguided brain. If he rejects it he does so only in the sense in which theologians have misrepresented freedom of will as being exempt from the law of causation. And in doing so he is certainly right in the face of Miss Bussey or any one who believes in a freedom of that kind, proclaiming that it is independent of causation.

There is no need of entering into the details of Miss Bussey's discussion. Any of our readers who knows Haeckel will be able to form his own judgment. Only a few points shall be mentioned here.

The universe has certainly to be explained from the highest

UMI

product its development achieves and not from its lowest beginnings. It is man that gives us the key to the appearance of the moner, while the moner will not be able to tell what its evolution will bring out in the end. On the other hand we have not solved the problem unless we trace the development of a rational being step by step in a mechanistic fashion of cause and effect. To deny it would mean to abolish science in spots. I prefer to keep my trust in science, for science to me is God's revelation. The most important step for instance is the development of reason, and it has been explained in a mechanistic sense by Ludwig Noiré when he shows how the origin of language has produced reason and not the reverse; or, to express his principle in a popular way, "We think because we speak" and not "we speak because we think." The mechanical mechanism of speech came first, and it was the mechanism of logic and grammar which has enabled us to think.

It is not a fault of Haeckel's if he holds the view that man explains the nature and significance of the moner. It proves that he is not onesided. His claim is but the natural consequence of a consideration of evolution.

The law of the conservation of matter and energy is an *a priori* law, which in its general meaning is similar to mathematical postulates. It is a demand of science and need not be proved in detail. It is a pre-supposition just as much as is the law of causation which the scientist assumes when he investigates natural phenomena. That there is a purpose in the universe is a proposition which would involve a belief that the universe as a whole is to be understood as an individual personal being after the fashion of a man. It would involve an anthropomorphic conception of God, and I doubt whether even among our theologians there are now many bold enough to take such a position. This, however, does not exclude that the universe in its processes follows a definite direction, a claim which is proved by the facts of evolution and is probably not denied by either a theistic or atheistic interpretation of the word.

Why the formulation of the theory of mechanicalism should be a fact which mechanicalism itself fails to explain is unintelligible, and why its own existence should be evidence of its own inadequacy is hard to understand, unless the notion of mechanicalism be narrowed to a limited field which does not include the entire construc-

tion of mechanicalism and its internal interrelations, such as for instance the interrelations of logical rules and conditions.

We may be able to uphold the theory of free will but we shall certainly not be able to deny the principle of determinism, and this is a blessing for the ethicist who preaches morality and claims that the freedom of will is essential for it, because if free will were indeed an exception to the law of causation and the will were undetermined and not changeable by education but remained a *tabula rasa* in spite of all attempts to change and improve it, or make it definite in the right direction, what would be the use of wasting our energies in promoting the welfare of mankind and eliminating evil influences? Let us be glad that determinism is true, for otherwise there would be no science, and principles of conduct would be a meaningless play of a misguided and erring imagination.

Haeckel apparently commits a very grave mistake. His opinions are "the result of the general presuppositions and prejudices of the age." He and many others "believe whatever fits in with their view of life and dismiss without a hearing anything which conflicts with it." Miss Bussey claims that "in this age of science the scientist has become the arbiter of all questions, and his view is commonly accepted as authoritative." In other words, we expect that science shall solve our problems, and we are prejudiced enough to bow down before science and accept its verdict. Haeckel for instance is so prejudiced that he believes in the universality of natural laws, and, says Miss Bussey, "It is a philosophic commonplace that laws resting on experience can be universalized only by means of the supposition of the uniformity of nature." It is a pity that Haeckel follows this fallacy and accepts the uniformity of nature, but the worst is that I too plead guilty. I believe not only in his "supposition of the uniformity of nature," but also in science with all that it implies, especially determinism which demands the determinedness of everything, even the determinedness of an unhampered and, in this sense, free will. I can not help it. I am in the same predicament as Professor Haeckel. May God have mercy on our souls! EDITOR.

THE BELIEF IN GOD AND IMMORTALITY.

Professor James H. Leuba, professor psychology and pedagogy in Bryn Mawr College, has undertaken to write a book on *The Belief in God and Immortality*. It is not a proof or disproof

of the doctrines essential in all positive religious creeds but a study of psychological statistics as to frequency and distribution of beliefs in a personal God and a personal immortality, and he finds that upon the whole in each group investigated as to their religious beliefs, the more distinguished fraction includes by far the smaller numbers of believers.

Professor Leuba's work is divided into three parts. The first part enters into a discussion of the characteristics of a belief in a continuation after death. He begins with the savage's idea of soul and ghost, setting forth in his second chapter the origin of the ghost idea, the appearance of ghosts in dreams and visions. He distinguishes from the belief in soul-ghosts the belief in immortality which he regards as late in the development of mankind. The fourth chapter is devoted to "The Origin of the Modern Conception of Immortality," beginning with a "translation to a land of immortality." The fifth chapter enters into metaphysics, the deductions of which however are regarded as insufficient.

In later days more scientific methods have been used by relying on physical and psychical manifestations and on the historical facts on which the resurrection of Christ is taught.

In Part II the belief in the personality of God and immortality is made an object of statistical study, first (Chapter VII) among the students of American colleges. In this it has become necessary to make a distinction between the personal and impersonal conceptions of God. The eighth chapter is devoted to an investigation of the belief in immortality, including a comparison of the changes taking place during college years. Here follows a detailed investigation (introduced first by the causes of the failure to answer and the interpretation of the questionnaire) of the beliefs held by the scientists, the historians, the sociologists, the psychologists, and the philosophers, concluding with a comparison of the signed and unsigned answers. He comments on the results of his investigation thus:

"The essential problem facing organized Christianity is constituted by the wide-spread rejection of its two fundamental dogmas—a rejection apparently destined to extend parallel with the diffusion of knowledge and the moral qualities that make for eminence in scholarly pursuits."

The third part which might be considered as independent of the first two is devoted to the question of the utility of the belief

in personal immortality and a personal God. Professor Leuba asks the question, "Is humanity better off with than without that belief (in a personal God and a personal immortality)? He answers: "The utility of the belief in immortality to civilized nations is much more limited than is commonly supposed.... we may even be brought to conclude that its disappearance from among the most civilized nations would be, on the whole, a gain."

It is noteworthy that his results show that the desire for immortality and the usefulness of the belief is rather disappearing with an increase of intelligence. There is an increasing tendency to disclaim any desire for immortality. This is in strong contrast to the supposition formerly quite common that even disbelievers yearn for immortality, but among the answers received to a questionnaire Professor Leuba finds even a relatively considerable number who abhor the idea of an endless continuation and he quotes a number of instances. For instance a woman thirty years of age declares that she has always felt death to be better than all else, anticipating it as the best thing life has to offer; and concluding with the sentence that death itself is a consummation devoutly to be wished.

Another letter is quoted as stating, "I feel a great dread of the possibility of having to live forever, or even again," and Professor Leuba quotes from Swinburne's poem "The Garden of Proserpina" the poet's hope of annihilation, where he says:

"Then star nor sun shall waken,
Nor any change of light;
Nor sound of waters shaken,
Nor any sound or sight;
Nor wintry leaves nor vernal,
Nor days nor things diurnal;
Only the sleep eternal
In an eternal night."

John Addington Symonds echoes the same ideas in prose. He says:

"Until that immortality of the individual is irrefragably demonstrated, the sweet, the immeasurably precious hope of ending with this life, the ache and languor of existence, remains open to burdened human personalities."

The greater stimulus for a desire for immortality comes in cases of the death of beloved persons, and the most impressive instance of this kind is quoted by Professor Leuba in the case of

a widow writing to her friend, the famous Professor Schleiermacher. Quoting from Schleiermacher's *Leben* as quoted by James Martineau in *A Study of Religion,* Vol. II, page 337:

"O Schleier, in the midst of my sorrow there are yet blessed moments when I vividly feel what a love ours was, and that surely this love is eternal, and it is impossible that God can destroy it; for God himself is love. I bear this life while nature will; for I have still work to do for the children, his and mine; but O God! with what longings, what foreshadowings of unutterable blessedness, do I gaze across into that world where he lives! What joy for me to die!

"Schleier, shall I not find him again? O my God! I implore you, Schleier, by all that is dear to God and sacred, give me, if you can, the certain assurance of finding and knowing him again. Tell me your inmost faith on this, dear Schleier; Oh! if it fails, I am undone. It is for this that I live, for this that I submissively and quietly endure: this is the one only outlook that sheds a light on my dark life,—to find him again, to live for him again. O God! he cannot be destroyed!"

In commenting that the psychological state might have been quite different in Schleiermacher's friend if she had remarried. Professor Leuba says: "In that occurrence her former yearnings for another life might have been replaced by dread of the time when she would be face to face with two husbands."

Perhaps the most dignified expression of an impersonal immortality has been expressed by George Eliot in her "Choir Invisible," but the main and classical instance is the orthodox Buddhist faith, and Professor Leuba quotes at length the text from Buddhist scriptures as translated by Henry Clarke Warren, where Buddha insists on not being born again and that the present life is his final entry into Nirvana. It reads thus:

"And being, O priests, myself subject to birth, I perceived the wretchedness of what is subject to birth, and craving the incomparable security of a Nirvana free from birth, I attained the incomparable security of a Nirvana free from birth; myself subject to old age,.... disease,.... death,.... sorrow,.... corruption, I perceived the wretchedness of what is subject to corruption, and, craving the incomparable security of a Nirvana free from corruption, I attained the incomparable security of a Nirvana free from corruption. And the knowledge and the insight sprang up within me, 'My

deliverance is unshakable; this is my last existence; no more shall I be born again.' And it occurred to me, O priests, as follows:

"'This doctrine to which I have attained is profound, recondite, and difficult of comprehension, good, excellent, and not reached by mere reasoning, subtile, and intelligible only to the wise. Mankind, on the other hand, is captivated, entranced, held spell-bound by its lusts; and forasmuch as mankind is captivated, entranced, held spell-bound by its lusts, it is hard for them....to understand how all the constituents of being may be made to subside, all the substrata of being be relinquished, and desire be made to vanish, and absence of passion, cessation, and Nirvana be attained.'"

It is peculiar that among scientists there was one who clung with great insistence to the belief in immortality, and this is no less an authority than the great biologist, Henri Pasteur, and he kept his religious faith and science in two different departments of his mind. He says:

"My philosophy is of the heart and not of the mind, and I give myself up, for instance, to those feelings about eternity which come naturally at the bedside of a cherished child drawing its last breath.

"There are two men in each one of us: the scientist, he who starts with a clear field and desires to rise to the knowledge of Nature through observation, experimentation, and reasoning; and the man of sentiment, the man of belief, the man who mourns his dead children and who cannot, alas, prove that he will see them again, but who believes that he will, and lives in that hope;.... the man who feels that the force that is within him cannot die."

Professor Leuba adds the following comment on Pasteur:

"I may remark incidentally upon the off-hand manner in which Pasteur divides life into two spheres, that of science and that of feeling, and apparently finds no use for logic and reason in the latter. This is a shocking example of a dangerous practice which, when carried to its logical consequence, would permit one to believe whatever he pleases. When I attempt to understand this attitude in a distinguished man of science, I can only conjecture that he treated religion as something primarily intended to comfort *anyway, anyhow.*"

Professor Leuba's book does not decide the question of the acceptability or unacceptability of the belief itself, but is simply a statistical investigation and for that reason possesses virtue for

theists as well as unbelievers in helping to find out the psychological state of things as it happens to be in our present generation, and from that standpoint the book will retain its virtue whatever be the position of the reader.

SIR OLIVER LODGE ON LIFE AFTER DEATH.

Sir Oliver has always been a believer in mediumistic experience and in the spirit existence of man in the other world, and in spite of his knowledge of physics he has taken a broad stand by coming out squarely and unreservedly in showing his faith. Details of such an expression might be amusing if it were not actually sad to see a man of his significance stooping to views which otherwise prevail only in the circles of half-educated people. His son Raymond died at the front in Flanders on September 14, 1915, and the bereaved father has published a book[1] containing a summary of his own philosophical views and a record of communications received from Raymond since his death.

From this we learn that Raymond woke up in the other world and got accustomed to his new surroundings. There are seven spheres all above the earth and turning around with the earth, but there is no consecutive night and day. It is always daylight except when one desires darkness; then night spreads according to one's wishes. Raymond resides in a house which appears to be made of brick, and spirit houses form streets in which the spirits walk and move. People who have lost arms or legs develop new ones as if by a kind of natural recuperation, so he tells his parents that he has replaced a tooth, and comrades of his who had lost arms or other limbs are restored to their original natural shape, but this restoration is not quite simple and there is a kind of spirit-doctors who help with their restoration. There is a special difficulty in restoring the spiritual body if the material body has been destroyed before its regeneration in the spirit world, so Raymond gives a definite warning that dead bodies should not be cremated before father has published a book containing a summary of his own they have been restored in the spirit plane of life.

The seven spheres which are built around the earthly plane seem to revolve with it at different rates of speed, so that the first sphere is not revolving at the same rate as the second, third, fourth, fifth, sixth and seventh spheres. Greater circumference makes the

revolution more rapid and this increase of rotation has an actual effect on the atmospheric conditions prevailing in different spheres. When asked for details about the nature of the other world Raymond said:

"What I am worrying about is how it is all made and of what it is composed. I have not found out yet, but I have a theory. It is not an original idea of mine. I was helped to it by words dropped here and there. People who think everything is created by thought are wrong. I imagined for a little while that one's thoughts over here formed the buildings and flowers and trees and solid ground; but there is something more than that.

"There is something always rising from the earth—something chemical in form. As it rises to ours it goes through various changes and solidifies here. I feel sure it is something given off from the earth that makes the solid trees, flowers, etc.....

"All the decay that goes on on the earth is not lost. It doesn't just form manure or dust. Certain vegetable and decayed tissue does form manure for a time, but it gives off an essence or a gas which ascends and which becomes what you call a 'smell.' Everything dead has a smell, if you notice; and I know now that the smell is of actual use, because it is from that smell that we are able to produce duplicates of whatever form it had before it became a smell. Even old wood has a smell different from new wood; you may have to have a keen nose to detect it on the earth plane.

"Old rags, cloth decaying and going rotten, all have smells. Different kinds of cloth give off different smells. You can understand how all this interests me. Apparently, so far as I can gather, the rotting wool appears to be used for making things like tweeds on our side. But I know that I am jumping; I'm guessing at it. My suit, I expect, was made from decayed worsted on your side.

"Some people here won't grasp this even yet—about the material cause of all these things. They go talking about spiritual robes made of light, built by thoughts on the earth plane. I don't believe it. They go about thinking that it is a thought robe they're wearing, resulting from the spiritual life they led; and when we try to tell them it is manufactured out of materials they don't believe it. They say, 'No, no; it's a robe of light and brightness which I manufactured by thought.' So we just leave it. But I don't say that they don't get robes quicker when they have led spiritual lives down

there; I think they do, and that's what makes them think that they made the robes by their lives.

"You know flowers how they decay. We have got flowers here; your decayed flowers flower again with us—beautiful flowers."

They have not only spirit doctors but also manufacturers and can provide you with materials if you so desire. Raymond himself does not smoke, but a friend of his, a great smoker on the earth plane, demanded cigars and he got them, but only about five; and the things given him looked like cigars, but after smoking about five cigars he no longer cared for more. He changed his habit and got accustomed to a more spiritual mode of life.

Colors have their significance, and different colors have different effects upon the character of the spirits.

"There's plenty of flowers growing here, you will be glad to hear. But we don't cut them here. They don't die and grow again; they seem to renew themselves. Just like people, they are there all the time renewing their spirit bodies. The higher the sphere he went to, the lighter the bodies seemed to be—he means the fairer, lighter in color. He's got an idea that the reason why people have drawn angels with long fair hair and very fair complexions is that they have been inspired by somebody from very high spheres."

The information Professor Lodge publishes was received from the medium Mrs. Leonard through her "control" known as "Fedo."

Incidentally we find a personal remark put in brackets and in italics of which Sir Oliver is apparently the authority. It reads: "A good deal of this struck me as nonsense, as if Feda has picked it up from some sitter."

Mediums have said much nonsense in print as well as in private seances, and the spirits of dead people have distinguished themselves by silly utterances; but the recent story of Raymond's communications rather excels all prior tales of mediumistic lore in the silliness of its revelations. But the saddest part of it consists in the fact that a great scientist, no less a one than Sir Oliver Lodge, has published the book and so stands sponsor for it.

Sir Oliver Lodge is a scientist who has done much creditable work and has written a number of books which exhibit keen thought and a good grasp of his subject, his specialty being physics. The books he has written are as folows:

Elementary Mechanics; Modern Views of Electricity; Pioneers of Science; Signalling Without Wires; Lightning Conductors and

Lightning Guards; School Teaching and School Reform; Mathematics for Parents and Teachers; Life and Matter; Electrons; Modern Views of Matter; The Substance of Faith; Man and the Universe; The Ether of Space; The Survival of Man; Parent and Child; Reason and Belief; and *Modern Problems.*

CURRENT PERIODICALS.

In Vol. XIV (1915) of the fifth series of the *Atti* of the Royal Academy of the Lincei at Rome is a publication in full of the treatise *De corporibus regularibus* of Pietro Franceschi or Della Francesca which was found in 1912 in the Vatican Library by G. Mancini. To this is prefixed a learned dissertation by Mancini to show that this treatise was pilfered by Luca Pacioli in his work on mensuration, the *Divina proportione*; and a report by Gino Loria on Mancini's memoir.

* * *

The articles of greatest interest to philosophical mathematicians in recent numbers of Vol. XVII (1916) of the *Transactions of the American Mathematical Society* are as follows. In the number for April, Robert L. Moore gives three systems of axioms for plane *analysis situs*—the non-metrical part of the theory of plane sets of points, including the theory of plane curves; Charles N. Haskins writes on the measurable bounds and the distribution of functional values of "summable" functions—which here means functions which are integrable in the generalized sense of Lebesgue; and Dunham Jackson proves in another way an important theorem of Haskins. In the number for July, L. L. Silverman discusses the generalization of the notion of the summability of a series to the limit of a function of a continuous variable; G. H. Hardy develops a new and powerful method for the discussion of Weierstrass's continuous function which is not differentiable, and allied questions; and William F. Osgood, to show that a theorem of Weierstrass for analytic functions of n complex variables is true for other "spaces" than that of analysis, lays down a general definition of "infinite regions," which includes the cases of projective geometry, the geometry of inversion, the geometry of the space of analysis, and so on.

* * *

In the *Bulletin of the American Mathematical Society* for June, 1916, Dr. A. Bernstein reduces the number of postulates which

Huntington gave in 1904 for Boole's algebra of logic from ten to eight, and that of postulated special elements from three ("zero", the "whole," and the "negative") to one (the "negative"). An interesting and valuable address delivered before the University of Chicago by Prof. Edward B. Van Vleck on "Current Tendencies of Mathematical Research" is printed in the October number.

* * *

The number of the *Revue de métaphysique et de morale* for May, 1916, contains a long and important article by A. N. Whitehead on the relationist theory of space. This theory is developed for a great part by help of the symbols of the author and Russell's work. The other articles in this number are by F. Colonna d'Istria (religion according to Cabanis), Léon Brunschvicg (the relations of the intellectual and the moral conscience), R. Hubert (the Cartesian theory of enumeration: on the fourth Rule of the *Discours*), and Georges Guy-Grand (impartiality and neutrality). In the July number of the *Revue* Lionel Dauriac writes on contingence and category, and tries to decide whether Kant was right or wrong in not separating the necessary and the *a priori*. Gaston Milhaud discusses the famous mystical crisis through which Descartes passed in 1619. Henri Dufumier maintains that the algebra of classes in logic only takes a systematic form if we consider it as a generalization of the mathematical theory of aggregates. F. Buisson explains "the true meaning of the sacred union." Finally, there is a necrology of Victor Delbos (1862-1916).

* * *

In the eighteenth volume (1916) of Prof. Gina Loria's quarterly *Bollettino di bibliografia e storia delle scienze matematiche*, the most interesting articles in the first two numbers (April and June) seem to be: J. H. Graf's collection of the correspondence between Ludwig Schläfli and some of his Italian mathematical contemporaries (pp. 21-35, 49-64); and G. Vivanti's review of the late Julius König's *Neue Grundlagen der Logik, Arithmetik und Mengenlehre* of 1914 (pp. 37-39).